ON THE FINITE ORDER OF WHITTAKER
FUNCTIONS, EISENSTEIN SERIES, AND
AUTOMORPHIC L-FUNCTIONS

MARK ELDON MCKEE

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Abstract

The theory of automorphic forms has seen an important breakthrough in the past few years. Shahidi and Kim [15] [16] [14], have been able to prove a few new cases of functoriality. One case of particular interest, is that they established the functoriality of $\text{Sym}^3 : GL_2 \to GL_4$. Further, these new results lead to the best estimates towards Selberg’s conjecture for $GL_2$ Maass forms. Of course, central to these new developments is the Langlands-Shahidi method, coming from the theory of cuspidally induced Eisenstein series.

The proof of functoriality in these cases requires the use of a converse theorem. For this, one must verify analytic information about $L$-functions related to the candidate functorial representation. One condition, is boundedness in vertical strips of the complex plane. The paper of Gelbart and Shahidi [7] proves this. They employ the theory of Eisenstein series. However, the proof of meromorphic continuation of Eisenstein series due to Langlands [20] requires the use of a functional equation in a somewhat analytically delicate way. Thus, so does the result in [7]. Further, they need to quote Müller’s estimates [23].

There is an ingenious Fredholm theory proof of meromorphic continuation due to Selberg where one does not require a functional equation. This thesis is a first attempt to make Selberg’s proof effective for the purpose of obtaining a self-contained result of the following form. For our group $G$, the Eisenstein series can be written as a ratio $F(s,g)/D(s)$, where $D$ and $F$ are entire of finite order, with the bounds for $F$ depending on $g$ in a compact set of $G$. We establish this result, but with the additional restriction that the cusp form have trivial $K$-type (so as to use spherical inversion). At the present time, we have also to rely on Müller’s estimates. We also prove a finite order result on Whittaker functions, the remaining local issue. In a handful of examples, we show how the main result combined with this local result gives the result of [7], without the delicate appeal to a functional equation.
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Chapter 1

Introduction

In recent years, a major breakthrough has been made in the theory of automorphic forms. Shahidi and Kim have been able to prove a few cases of functoriality, part of a program conjectured by Langlands.

At present, to understand certain cases of functoriality, it is necessary to study the Langlands-Shahidi method. Essentially, this is the theory of relating analytic information about Eisenstein series to automorphic $L$-functions (cf. Langlands [19], [20]). More accurately, for a linear algebraic group $G$ over the adele ring of a number field, and an automorphic cuspidal representation $\pi$ on the Levi factor of a maximal parabolic subgroup, an Eisenstein series, $E(s, \pi, g)$ attached to $\pi$ can be created. This is a spectral object for $g \in G$, and holomorphic for $\Re s$ sufficiently large. $L$-functions related to this cusp form appear in the Fourier coefficients (along a unipotent subgroup) of the Eisenstein series.

In more detail, suppose $M$ is the above Levi, and $N$ is the unipotent subgroup of the above maximal parabolic. Then $M$ acts on $N$ by the adjoint representation. Let $r$ be the corresponding action of the dual of $M$ on the dual of the complexification of the Lie algebra of $N$. Further, $\pi = \otimes_v \pi_v$, and each unramified $\pi_v$ gives rise to an eigenfunction under convolution of the local Hecke algebra. The automorphic
$L$-functions that appear in a constant term of $E(s, \pi, g)$ are related to these Hecke eigenvalues and to the decomposition of $r$. This is the work of Langlands [19], where we first meet the definition of the dual group of $G$, the $L$-group, $^LG$. The known meromorphic properties, such as meromorphic continuation and functional equation (cf. Langlands [20]), and location of poles of the Eisenstein series then give information about these $L$-functions. For example, for $\pi$ a cusp form on $GL_2$, Kim and Shahidi [15] used this method to prove the holomorphy of $L(s, Sym^3(\pi))$. (In this case, $GL_2$ is a Levi component of an exceptional group of type $G_2$. Significant representation theoretic information is also needed in this case, such as the classification of the unitary dual of a $p$-adic group of type $G_2$ due to Muić [22].)

Suppose $r$ decomposes as $\oplus_i r_i$. Then each $r_i$ is a homomorphism;

$$r_i : ^LM(\mathbb{C}) \to GL_{n_i}(\mathbb{C})$$

for some $n_i$. Suppose we have a homomorphism $\phi : ^LG(\mathbb{C}) \to ^LH(\mathbb{C})$ for two groups $G$ and $H$. The statement $\phi$ is functorial then means $\phi$ lifts automorphic cuspidal representations on $G$ to automorphic representations on $H$. In [16], Kim and Shahidi proved

$$GL_2 \times GL_3 \to GL_6$$

is functorial. From this, they were able to establish $Sym^3 : GL_2 \to GL_4$ is functorial, and they determined when the image is cuspidal, a truly striking result. There are many technical complications, but the heart of the matter lies in the Langlands-Shahidi method. See Kim and Shahidi [17] as well as Kim [14] for new results, and many new applications. For example, this gives the best estimate towards Selberg’s conjecture for $GL_2$ Maass forms.

For the proof of these significant cases of functoriality, one has to apply a con-
verse theorem (cf. Cogdell and Piatetski-Shapiro [6]). To do this, one has to verify certain analytic information about related $L$-functions. More specifically, suppose $\rho$ is the candidate functorial representation. Let $\tau_n$ denote a cuspidal automorphic representation of $GL_n$. One has to consider $L$-functions of $\rho$ twisted by various $\tau_n$ for $n$ up to $\dim(\rho) - 2$. In the paper [16], this is the reason for the appearance of exceptional groups. (Of course there are large technical difficulties, but this is the heart of the matter.) One then has to verify certain analytic conditions about these $L$-functions. One of these conditions is that they must be bounded in vertical strips (of finite width). The paper of Gelbart and Shahidi [7] proves this. They conclude this by proving these $L$-functions are of finite order, employing the theory of Eisenstein series.

This, essentially is a problem in making effective the proof of the meromorphic continuation of Eisenstein series. More precisely, finite order estimates are needed from an effective proof. However, the proof of meromorphic continuation by Langlands [20] does not give much information beyond meromorphy. So, they (Gelbart and Shahidi [7]) need to bring in some elaborate estimates and machinery. They start by using Müller’s estimates [23] for the intertwining operator, found in a constant term of $E(s, \pi, g)$. Using Müller, and a Maass-Selberg type relation (Langlands’ inner product formula), they show a non-constant Fourier coefficient is a ratio of two functions of finite order in any strip of finite width in $\Re s \geq 0$. Then, using induction, and a local result on the finite order of Whittaker functions in $\Re s \geq 0$, they are able to conclude such an $L$-function is of finite order in any vertical strip in $\Re s \geq 1/2$. They same will be true in $\Re s \leq 1/2$, by the functional equation. However, the bound deteriorates as $\Re s \to 1/2$. Under a local assumption on $\pi$, they show that $L(s, \pi, r_i)$ can have only a finite number of poles in the plane. If $L_0(s, \pi, r_i)$ denotes the function after poles have been cleared, the specific bound is of the form $\log |L_0(s, \pi, r_i)| \leq \frac{C}{\cos \theta}(r^{\beta} + 1)$ if $s - 1/2 = re^{i\theta}$, for some $\beta > 1$ and constant $C$. Then they appeal to Matsaev’s
Theorem (Lecture 26 in Levin [21]), a delicate complex analysis result to conclude $L_0(s, \pi, r_i)$ is entire of finite order. In the above, a Whittaker function is a local integral coming from a non-zero Fourier coefficient.

Sarnak suggested that there should be a direct proof; one that doesn’t need Müller, Matsaev, or any functional equations. There is an ingenious proof due to Selberg (outlined in his Collected Works [27]) which, using the spectral properties of the construction of the Eisenstein series, reduces the continuation to an application of Fredholm integral equation theory. (See the notes on this proof by Cohen and Sarnak [5], or the detailed account of Wong [35].) What separates Selberg’s Fredholm determinant proof from Langlands [20] or Harish-Chandra [10], is that one can meromorphically continue the Eisenstein series to any compact region in the plane without ever appealing to a functional equation. This includes crossing the unitary axis, where the other proofs become somewhat delicate. One would expect that Selberg’s proof could be made effective, and would give the Eisenstein series $E(s, \pi, g)$ could be written as $F(s, g)/D(s)$, where $F$ and $D$ are entire, finite order functions in the plane, with the bounds for $F$ depending smoothly for $g$ in a compact set of $G$.

The main result of this thesis establishes this result, but under the additional assumption that $\pi$ has trivial $K$-type. Further, we do not appeal to Matsaev’s theorem, or any functional equation in any delicate way. In our present treatment, we still employ Müller’s estimates. We avoid many difficulties in Selberg’s proof by inverting the Selberg transform (that is why we need to assume trivial $K$-type), to find a compactly supported point-pair invariant which is not smooth. It is the author’s speculation that Selberg also saw the same difficulties, since both of these issues, non-smooth kernels and inverting the basic spherical transform, are addressed in Selberg’s original work [26]. It is the author’s enthusiastic hope to remove the dependence on Müller’s estimates, and spherical inversion to establish this result in a completely self-contained way.
Crucial to conclude the finite order result for these Langlands-Shahidi $L$-functions, is that we need a finite order bound for Whittaker functions in the entire plane. This is the remaining local issue, especially at the Archimedean places. We have established this by making effective the proof of holomorphy of these functions due to Jacquet [13]. Combined, this gives a proof that these global $L$-functions are of finite order without appealing to a functional equation in any analytically delicate way. However, we have limited our discussion to just a handful of examples.

Let us describe the contents of this thesis. First of all, in using the Langlands-Shahidi method, one assumes the cusp form $\pi$ on the Levi is generic. That is, translating by an element of the full unipotent subgroup of $M$, the representation pulls out an additive character of this subgroup, say $\psi$. We assume $\psi$ extends to an additive character on the full unipotent subgroup $U$ of $G$, which is as nontrivial as possible. Then, taking the (adelic) $\psi$-Fourier coefficient of the Eisenstein series built from $\pi$, we get an inverse product of partial $L$-functions times a finite number of one dimensional Whittaker functions (cf. Shahidi [30] [32]). These Whittaker functions are the same as those of Jacquet [13] restricted to an appropriate line in $\mathbb{C}^n$. In Chapter 2 we obtain a $\mathbb{C}^n$-finite order estimate for these functions. We do this by following Jacquet’s ([13]) proof of holomorphy very closely, and using convexity in $\mathbb{C}^n$.

Chapter 3 involves the theory of the spherical (Selberg) transform. Suppose that $K$ is the maximal compact subgroup of our real or complex group $G$. Let $D(G)$ denote the left invariant differential operators, and $D_K(G)$ the subset which is invariant on the right by $Ad(K)$. It is known that all functions which are bi-invariant under $K$, and which are eigenfunctions of $D_K(G)$ have a very simple form, and are parametrized by $\lambda \in \mathbb{C}^n$, up to the action of the Weyl group. Let $\phi_\lambda$ denote such a function. If $f$ is a smooth, compactly supported point-pair invariant on $G$, we set $\tilde{f}(\lambda)$ to be convolution of $f$ with $\phi_\lambda$ evaluated at the identity element of $G$. This is the spherical transform of $f$, and is invariant under the Weyl group. Further, since $f$ has compact support,
and is smooth, $\tilde{f}(\lambda)$ will satisfy a Paley-Weiner condition, but will otherwise be of rapid decay. A good reference for the theory of the spherical transform is Helgason [12]. It is known that this transform can be inverted. Chapter 3 is essentially just a density argument to show if we start with such a transform which doesn’t have rapid decay, we can still invert to a compactly supported function which is not smooth.

As usual in this theory, Selberg’s Fredholm proof of continuation of Eisenstein series involves cut-off operators, and constant terms. We know the form of the constant terms of the Eisenstein series, up to the intertwining operators. Selberg’s proof inverts this information, using the theory of compact operators, to recover the Eisenstein series. Of course the Selberg transform of a point-pair invariant, a smooth, compactly supported, bi-$K$ invariant function is central. Chapter 4 is just a preparatory chapter to show that if we start with a compactly supported point-pair invariant which is not smooth, the relevant operators are still compact. We also obtain a finite order estimate for the denominator of the resolvent of these operators.

Chapter 5 is the bulk of this thesis. In section 5.1, we define our Eisenstein series, which are constructed from an automorphic cusp form $\Phi$ on the Levi factor $M$ of a maximal parabolic subgroup $P$ of $G$. So, we assume there is a discrete subgroup $\Gamma$ of $G$. Further, we assume $\Phi$ has trivial $K$-type. We list a few simple properties, and discuss the intertwining operators. Since $P$ is maximal, the complex parameter $\lambda \in \mathbb{C}$. $E(\lambda, \Phi, g)$ will denote our Eisenstein series. In section 5.2, we discuss the numerator of these resolvent operators. We obtain a finite order estimate for these numerators for $\lambda$ in any strip of finite width with center the imaginary axis, uniformly for $g \in \Gamma \backslash G$. This estimate is conditional; if $\alpha$ is the point-pair invariant, this estimate is assuming the inverse of the Selberg transform of $\alpha$ has polynomial growth in the strip. In section 5.3, by inverting the spherical transform, we produce a compactly supported point pair invariant whose inverse Selberg transform has polynomial growth, and is nonzero in the strip. If $F^*_f(\lambda, g)$ denotes the Fredholm solution of the appropriate
resolvent, we discuss the analytic properties (with effective estimates) for $\lambda$ in a strip, and $g$ in a compact set of $G$. We see $E(\lambda, \Phi, g) = \sum_J \theta(J, \lambda) F_J^*(\lambda, g)$ in the strip, where the $\theta(J, \lambda)$ are matrix coefficients of intertwining operators. In section 5.4 we explain how to recover the $\theta(J, \lambda)$ from the $F_J^*(\lambda, g)$ evaluated at a finite number of points on the group; this is the uniqueness principle of Selberg. We show how this gives each $\theta(J, \lambda)$ is a ratio of two holomorphic functions with finite order estimates in the strip. Unfortunately, this is not enough analytic information to pull out a finite order Weierstrass product in the strip containing the poles of $\theta(J, \lambda)$.

In section 5.5, this is exactly solved by appealing to Müller [23]. However, we have $E(\lambda, \Phi, g) = \sum_J \theta(J, \lambda) F_J^*(\lambda, g)$, and so to keep track of the poles of $E$ for $\lambda$ in the strip, we need a finite order type of estimate for the number of poles of $F_J^*(\lambda, g)$. The theory of Fredholm determinants gives enough analytic information for us to do this. This takes a little of analysis with our specific transform. We now take a strip, symmetric about the imaginary axis, and large enough to overlap the region of absolute convergence of $E$. From above, we have bounds for $E$ in the strip. If we stay in the half plane of absolute convergence, but uniformly away from the line of convergence, we quote Harish-Chandra [10] for a trivial finite order estimate to this region. We now combine Müller with the functional equation and the estimates from absolute convergence to conclude effective estimates for the reflection of this region through the imaginary axis. (We still stay uniformly away from the reflected line.)

We have used the functional equation, but in a basic way. Since we can choose the strip to have a large overlap with these half planes, we conclude the main result of this work, Theorem 2. $E(\lambda, \Phi, g)$ can be written as $F(s, g)/D(s)$, where $D$ and $F$ are entire with computable finite order, and the bounds for $F$ depend smoothly for $g$ in a compact set of $G$. Sections 5.6 and 5.7 are small sections to ensure the main result is valid in the case of several cusps, and in the adelic situation.

Chapter 6 illustrates how Theorem 1 combined with the main result leads to
an easier proof of boundedness in vertical strips of these \( L \)-functions. However, we impose some restrictions on \( G \) for technical reasons. In this last chapter, we assume \( G \) is real, split over \( \mathbb{Q} \), and semi-simple which is either a classical group, or of type \( G_2 \). Further, we only give a handful of examples, so the inductive step (as in [7]) is far from complete. (Recall also that the \( K \)-type of the cusp form \( \pi \) must be trivial.) These examples do include the completed \( L \)-functions \( L(s, Sym^2(\pi)) \), \( L(s, \wedge^2(\pi)) \) if \( \pi \) is an automorphic cusp form on \( GL_n(\mathbb{A}_\mathbb{Q}) \), \( L(s, \pi_1 \times \pi_2) \) if \( \pi_1 \) and \( \pi_2 \) are automorphic cusp forms (with trivial central characters) on \( GL_n(\mathbb{A}_\mathbb{Q}) \) and \( GL_m(\mathbb{A}_\mathbb{Q}) \) respectively (i.e., Rankin-Selberg), and of course \( L(s, Sym^3(\pi)) \) for a \( GL_2(\mathbb{A}_\mathbb{Q}) \) cusp form.
Chapter 2

Whittaker functions

2.1 The proof of Jacquet

The Whittaker functions of Jacquet [13] are local integrals of a function in the principle series twisted by a local generic character. Further, they depend upon a complex parameter $\lambda \in \mathbb{C}^n$, where $n$ is the semi-simple rank of our group. In [13], Jacquet proves they are holomorphic in $\lambda$. Following this paper carefully, and using convexity in $\mathbb{C}^n$, we will make this proof effective, giving a $\mathbb{C}^n$-type finite order bound of at most $n$. Later, we will see these functions appearing in a non-constant term of an Eisenstein series (cf. equation (6.1)). Clearly, for our purposes, it is useful to have an effective global estimate for these Whittaker functions.

In the first half of this section, we only use the reference Jacquet [13]. So, unless otherwise stated, any proposition, theorem, section (§), or referencing the name Jacquet will refer to [13]. Further, we will use notation similar to [13]. This includes that actions are backwards of what they are today; for example, $K$ acts on the left. Clearly, this is only a difference in language, and will not change any basic result.

Let $G$ be a semi-simple Chevalley group, defined over a number field. In what follows, all calculations and integrals are local, over either $\mathbb{R}$ or $\mathbb{C}$. We can write
\( G = KAU \), with maximal compact \( K \), split component \( A \), and unipotent radical \( U \) consisting of all positive roots. \( V \) is the unipotent group opposite \( U \). We assume \( \chi \) is a character on \( A \) of modulus one, and \( \zeta \) is a nontrivial character on \( V \). In particular, \( \zeta \) must be nontrivial on each \( V_\alpha \) for each independent root \( \alpha \). \( D \) is a finite dimensional unitary representation of \( K \). \( P(D, \chi) \) is the orthogonal projection onto vectors whose \( K \)-type matches \( \chi \) on \( A \cap K \). Let \( B(V) \) denote the interior of the positive Weyl chamber. Finally, \( \lambda \), our complex multi-variable, will be in the dual of the complex lie algebra of \( A \). More specifically, for independent roots \( \{ \alpha_i \} \), we write \( \lambda = \sum s_i \Lambda_i \) with \( s_i \in \mathbb{C} \). The \( \Lambda_i \) are characterized by the relations \( \langle \Lambda_i, \alpha_j \rangle = 0 \) for \( i \neq j \), and \( \langle \Lambda_i, \alpha_i \rangle = \frac{2\langle \alpha_i, \alpha \rangle}{\alpha_i} \).

Set

\[
L(g, D, \chi, \lambda) = D(k)\chi(a) < -\lambda, a > P(D, \chi)
\]

where \( g = kau \). Now set

\[
E_V(g, D, \chi, \lambda; \zeta) = \int_V L(gv, D, \chi, \lambda + \rho)\zeta(v)dv
\]

where \( \rho \) is half the sum of positive roots of \( U \). With this assumption on \( \zeta \), Jacquet shows \( E_V \) is holomorphic in all of \( \mathbb{C}^n \). We will make this proof effective, showing that for \( g = e \), as a function of \( \lambda \in \mathbb{C}^n \), \( E_V \) is of finite order \( n \).

Define \( E_{V_\alpha}(g, D, \chi, \lambda; \zeta) \) to be the integral over \( V_\alpha \) of the same integrand. Here we assume \( \lambda \in B(V) \), so \( E_V \) (and consequently \( E_{V_\alpha} \)) converges absolutely (see §2). There is always an isomorphism \( SL_2 \leftrightarrow G^\alpha \), and essentially, the restriction of \( E_{V_\alpha} \) to \( g \in G^\alpha \) is an \( SL_2 \), or one variable problem. In particular, we have holomorphy, a functional equation, and a bound that lets us cross the \( \alpha \)-wall of \( B(V) \). (The normalization of \( \rho \) inside a one variable integral works exactly because \( s_{\alpha_i} \) is in the \( \Lambda_i \) direction, instead
of the $\alpha$ direction). Crucial to Jacquet, is that this is true for all $g \in G$. (Further, let us mention in the functional equations and estimates below, we assume the $K$-type $D$ restricted to $G^\alpha$ is irreducible. This is what Jacquet [13] does. Of course this will not be true in general. However, one can see, as Jacquet points out on page 255, that the general cases follows from this. One just has to keep track of different blocks of matrix coefficients.)

Before stating the functional equation for $E_{V_\alpha}$, let us mention we will explain all the relevant information ($\omega_\alpha$, the function $C(s_\alpha)$, and the operator $R_\alpha$) after the statement.

Lemma (3.2) of Jacquet: Let $\mu_\alpha \neq 0$ be determined by $\zeta$ on $V_\alpha$.

(ii) For $g \in G$, we have

$$E_{V_\alpha}(g, D, \chi, \lambda; \zeta) = \overline{\chi_\alpha(\mu_\alpha)} |\mu_\alpha|^s E_{V_\alpha}(g, D, \omega_\alpha \chi, \omega_\alpha \lambda; \zeta) R_\alpha(D, \chi_\alpha, s_\alpha).$$

(iii) For $b > 0$, there exists a constant $B$ so that

$$|E_{V_\alpha}(g, D, \chi, \lambda; \zeta)| \leq BC(s_\alpha) E_{V_\alpha}(g, \lambda_b)$$

for $|\Re s_\alpha| \leq b$ and $\lambda_b = \Re \lambda - \frac{1}{2}(\Re s_\alpha - b)\alpha$.

Now, up to the absolute value of a power of $\pi$ or $2\pi$, we can take $C(s) = \frac{c_{1,\alpha}}{|1(s/2 + c_2,\alpha)|}$, where $c_{1,\alpha}$ and $c_{2,\alpha}$ are constants depending on $\chi$ and the $K$-type, $D$ restricted to $G^\alpha$. Here, we are using the explicit computations of Jacquet (see §4, 5) for $L_\phi(\chi, s)$, for the privileged function $\phi$, over $\mathbb{R}$ or $\mathbb{C}$. (See §1 for $L_{\phi}$) $B$ and $C(s_\alpha)$ actually come from the maximum principle along with a privileged Schwartz function (see §1, 4, 5 of Jacquet). Further, $\omega_\alpha$ is the Weyl element corresponding to reflection by $\alpha$ and $R_\alpha$ is a one variable operator, that is meromorphic in $s$, and holomorphic if $\Re s < 1$. Later, we will be using the explicit computations of these operators by Jacquet. We will be concerned with a finite order bound of $R_\alpha$ in $\Re s \leq 0$. In complex multi-variable
coordinates, $\lambda_b$ is exactly a shift of $\Re \lambda$ in the $\alpha$ direction, until the $s_\alpha$ coordinate becomes $b$; i.e., a reflection through the $\alpha$-wall. $E_{V_\alpha}(g, \lambda_b)$ is the trivial upper bound for $E_{V_\alpha}$ obtained by taking absolute values inside the integral.

Let us sketch the proof of Proposition (3.3) of Jacquet. Basically, this is the statement that $E_V$ has the same functional equation as each $E_{V_\alpha}$. Fix $g \in G$. (Jacquet lets $g$ vary in a compact set for uniform convergence issues on $G$. We will not need this.) Let $V'$ be the subgroup of $V$ generated by all negative roots distinct from $-\alpha$. Then $V = V'V_\alpha$. For $\lambda \in B(V)$ we have absolute convergence, and we can write

$$E_V(g, D, \chi, \lambda; \zeta) = \int_{V'} E_{V_\alpha}(gv', D, \chi, \lambda)\zeta(v')dv', \,$$

convergence being uniform on compact sets. By Lemma 3.2, for $\lambda$ in a compact set such that $|\Re s_\alpha| \leq b$ and $\lambda_b \in B(V)$, we have

$$|E_{V_\alpha}(gv', D, \chi, \lambda)| \leq B' E_{V_\alpha}(gv', \lambda_b).$$

This integrated over $V'$ shows the above integral expression for $E_V(g, D, \chi, \lambda; \zeta)$ is dominated by

$$B' \int_{V'} E_{V_\alpha}(gv', \lambda_b)dv', \,$$

which converges uniformly on this compact set. This gives a holomorphic extension to $\omega_\alpha B(V)$, and a functional equation for $E_V(g, D, \chi, \lambda; \zeta)$ which is the same as the functional equation for $E_{V_\alpha}(g, D, \chi, \lambda; \zeta)$. Clearly, $B'$ can be taken to be $B \max \{C(s_\alpha)\}$, the maximum being taken over $\lambda$ in this compact set.

A slight variant of this will be suitable for our purposes. Fix $g = e$. Let $B_\epsilon(V)$ denote the closure of the set of points in $B(V)$ that are at least $\epsilon$ away from the walls of $B(V)$. Here, we are taking euclidean distance (which we will relate to the
coordinates \((s_1, \ldots, s_n) \in \mathbb{C}^n\) below. See the picture below (rank 2 sketched).

We know \(E_V(e, D, \chi, \lambda; \zeta)\) is actually bounded in \(B_\epsilon(V)\). This follows from the original integral expression for \(E_V\): taking absolute values inside the integral, \(D, \chi, \zeta\) all become trivial and \(\lambda\) becomes \(\Re \lambda\). Now, the integral factors into rank one intertwining integrals, and becomes a product of factors of the form \(\frac{\Gamma(t)}{\Gamma(t+1/2)}\) for \(t \gg \epsilon\).

Once again, let’s consider

\[
E_V(e, D, \chi, \lambda; \zeta) = \int_{V'} E_{V_\alpha}(ev', D, \chi, \lambda)\zeta(v')dv'.
\]

Take \(\lambda\) so that \(|\Re(s_\alpha)| \leq b\) and \(\lambda_b \in B_\epsilon(V)\). We know

\[
|E_{V_\alpha}(v', D, \chi, \lambda; \zeta)| \leq BC(s_\alpha)E_{V_\alpha}(v', \lambda_b).
\]

We can now use the same argument to show that the above integral is dominated pointwise by \(BC(s_\alpha)E_V(e, \lambda_b)\) for \(\lambda\) in such a region. (We still have uniform convergence on compact sets, but Lemma (3.2) gives the particular constant \(B\) for \(s_\alpha\) in the strip \(|\Re(s_\alpha)| \leq b\).) For each simple root \(\alpha\), we choose \(b\) so that \(\lambda_b\) is on the boundary of \(B_\epsilon(V)\) closest to the \(\alpha\)-wall. Let \(\tilde{B}_\epsilon(V)\) be the union of these regions with \(B_\epsilon(V)\). See the picture below (rank 2 sketched).
Fix $\beta \in \mathbb{R}$ with $\beta > 1$. For simplicity, let us denote by $\{s_i\}$ the point $(s_1, \ldots , s_n)$, and by $|\{s_i\}|$ its normal length in $\mathbb{C}^n$. We must now state some observations with distance. In these coordinates, $s_i$ and $s_j$ are not necessarily orthogonal. Suppose $\{e_i\}$ is an orthonormal basis of the complex dual of the Lie algebra of $A$. Then there is a fixed transition matrix which takes any point in its $\{s_i\}$ coordinates to its $\{e_i\}$ coordinates. From Cauchy-Schwarz, it follows that distance in the $\{s_i\}$ coordinates is equivalent to Euclidean distance. This is true for $s_i \in \mathbb{R}$, and then easily to $\mathbb{C}^n$.

Now $E_V$ is bounded in $B_\epsilon(V)$, and essentially up to a power of $\pi$ or $2\pi$, $C(s_\alpha)$ is the inverse of a gamma factor in $s_\alpha$. Geometrically, easily the euclidean length of $\lambda$ is less than or equal to the euclidean length of $\lambda_b$ in one of the regions we have added to $B_\epsilon(V)$ above. Thus, the same will be true, up to a fixed constant for these lengths in the $\{s_i\}$ coordinates. We will absorb this constant into the power $\beta$ below. Since $\frac{1}{\Gamma(s)}$ is of order one, we have easily

$$|E_V(e, D, \chi, \lambda; \zeta)| \ll \epsilon \exp(|\{s_i\}|^\beta)$$

for $\lambda \in \tilde{B}_\epsilon(V)$, the constant also depending on $\beta$, the constant in Lemma (3.2) (applied for each $\alpha_i$), and the constants in $C(s_\alpha)$.

Now we must look at the operator $R_\alpha(D, \chi_\alpha, s_\alpha)$ that appears in the functional equation of $E_V$. In §4, Jacquet computes this when our local field is $\mathbb{R}$. Up to a
constant, and a power of $\pi$,

$$R_\alpha(D, \chi_\alpha, s_\alpha) = \frac{\Gamma\left(\frac{1}{2}(1 - s_\alpha) + d_{1,\alpha}\right)}{\Gamma\left(\frac{1}{2}(1 + s_\alpha) + d_{2,\alpha}\right)}.$$ 

Here the $d_{i,\alpha}$ are constants that depend on the $K$-type restricted to $G^\alpha$, and $\chi_\alpha$. Crucial to us, is that $\Re(d_{i,\alpha}) \geq 0$. In §5, Jacquet computes $R_\alpha$ over $\mathbb{C}$. Up to projection operators, a constant, and a power of $2\pi$,

$$R_\alpha(D, \chi_\alpha, s_\alpha) = \frac{\Gamma\left(1 - s_\alpha + e_{1,\alpha}\right)}{\Gamma\left(1 + s_\alpha + e_{2,\alpha}\right)},$$

with similar remarks for the $e_{i,\alpha}$. Crucial to us, once again is that $\Re(e_{i,\alpha}) \geq 0$. We see in each case that

$$|R_\alpha(D, \chi_\alpha, s_\alpha)| \ll \exp \left(|\{s_i\}|^{\beta}\right)$$

in the half plane $\Re s_\alpha \leq 0$, with the constant depending on $D$ and $\chi_\alpha$. Here, $\alpha = \alpha_i$. 

In what follows, we will be interested in effective bounds. So, we assume holomorphy of $E_V$ on all of $\mathbb{C}^n$. This is Theorem (3.4) of Jacquet. Let us recall the functional equation for $E_V$. It is:

$$E_V(e, D, \chi; \lambda; \zeta) = \nabla_\alpha(\mu_\alpha)|\mu_\alpha|^{s_\alpha}E_V(e, D, \omega_\alpha \chi, \omega_\alpha \lambda; \zeta)R_\alpha(D, \chi_\alpha, s_\alpha).$$

Let us recall Lemma (3.4.2) of Jacquet: Let $\omega = \omega_\alpha \omega'$, where $l(\omega) = q + 1$, and $l(\omega') = q$. If $\lambda \in \omega B(V)$, then $\Re s_\alpha \leq 0$, where $\lambda = \sum s_i \Lambda_i$.

Let $B = B(V)$. Let $\lambda$ be in the interior of some Weyl chamber, not $B$. Then $\lambda \in \omega_\alpha \omega B$ for some element $\omega_\alpha \omega$, with $l(\omega_\alpha) = q + 1$, $l(\omega) = q$ for some $q \geq 0$. So, the above lemma shows $\Re s_\alpha \leq 0$. Suppose $E_V$ has a specific bound in $\omega B$. The functional equation shows it has a bound at $\lambda$ that depends on $\mu_\alpha$, the bound in $\omega B$, and the
bound for $R_\alpha$ in $\Re s_\alpha \leq 0$. (Actually, $\chi$ has been conjugated with this reflection, but this does not cause a problem for us.)

Let $M_\epsilon (V) = \bar{B}_\epsilon (V) \cap B(V)$. Clearly, the euclidean length of $\lambda$ and $\omega_\alpha \lambda$ are the same. It follows the lengths of these points are proportional in the $\{s_i\}$ coordinates, and so one is bounded by a fixed constant times the other. Once again, for each reflection, this constant can be absorbed into $\beta$. Using the above bounds for $E_V (e, D, \chi, \lambda; \zeta)$ in $M_\epsilon (V)$, and $R_\alpha$ for $\Re s_\alpha \leq 0$, the above argument shows we have

$$|E_V (e, D, \chi, \lambda; \zeta)| \ll \epsilon \exp (l |\{s_i\}|^\beta),$$

(2.1)

for $\lambda \in \cup \omega M_\epsilon (V)$, the constant depending on $\beta, D, \chi$ and $\zeta$. Here, $l$ is the length of the longest Weyl element. Let $Q_{i,j}$ be the intersection of the $\alpha_i$-wall and the $\alpha_j$-wall of $B(V)$. Let $M(V) = \cup \omega Q_{i,j}$, the union being over all $\omega$ and $Q_{i,j}$. Let $M_\epsilon$ denote $\cup \omega M_\epsilon (V)$. Then essentially, the complement of $M_\epsilon$ is an $\epsilon$ ball around $\cup \omega M(V)$, and equation (2.1) is valid in $M_\epsilon$.

For a several complex variables reference, we use the book of Bochner and Martin [1]. Denote by $C(z, r_j)$ the polydisc $\{s_j : |z_j - s_j| < r_j\}$ for $j = 1, \ldots, n$. We will state an easily proved adaptation of Theorem 10 from §4 of Chapter 5 of this book. Theorem Let $f(s_1, \ldots, s_n)$ be holomorphic in the $n$ polydiscs $C(z, r_{i,j})$, for $i = 1, \ldots, n$. Then $f$ has a holomorphic continuation to the union of all polydiscs of the form $C(z, r_{j,\theta})$ where

$$\log r_{j,\theta} = \sum_i \theta_i \log r_{i,j},$$

the continuation being effected by the Taylor series expansion for $f$ at $z$. Here, $\theta_i \geq 0$ and $\sum \theta_i = 1$.

(See §4 of Chapter 5 of [1]. This comes from mapping polystrips to these polydiscs.
$f$ then has a series expansion in $e^z$ in these polystrips, which converges absolutely in their interiors. Using the convexity of $e^x$, for $x \in \mathbb{R}$, the continuation comes from seeing this expansion converges absolutely for convex combinations of points in these polystrips.)

Let $\epsilon$ be fixed, and very small. We select a point $(a_1, \ldots, a_n)$ deep inside $B(V)$ with the following property. Let $B_2(a)$ denote the sphere of radius two around this point. (Distance taken in the $\{s_i\}$ coordinates.) Here, we are viewing $B(V) \subset \mathbb{R}^n$, so each $a_j \in \mathbb{R}$. The property we require is that $B_2(a)$ must stay inside $M_\epsilon$ for any shift of any length in each $\Lambda_i$ direction. Geometrically, this is possible, and rather crude, since we are only trying to avoid a finite number of co-dimension rank two hyperplanes. Let $l_1$ be the length (in the $\{s_i\}$ coordinates) of $(a_1, \ldots, a_n)$.

Fix $(z_1, \ldots, z_n) \in \mathbb{C}^n$. Let us assume this point is not in $M_\epsilon$. Let $l_2$ be the length of $(z_1, \ldots, z_n)$. Since $(z_1, \ldots, z_n)$ is fixed, let $\zeta$ be our $\mathbb{C}^n$ variable. Let us write a new coordinate $\{s_j\} = \{\zeta_j\} - \{a_j\}$. For convenience, let $f(\{s_j\}) = E_V(e, D, \chi, \{s_j\} + \{a_j\}; \zeta)$. Let $l_3 = (l_1 + l_2 + 1)^n$. Then, by the triangle inequality (in the $\{s_i\}$ coordinates), $(a_1, \ldots, a_n)$ and $(z_1, \ldots, z_n)$ are within $l_3^{1/n}$ of each other. Let $C(1,j)$ denote the polydisc $\{\zeta_i : |a_i - \zeta_i| < 1\}$ for $i \neq j$ and $\{\zeta_j : |a_j - \zeta_j| < 2^n l_3\}$. Similarly, let $C(2,2j)$ denote the polydisc $\{\zeta_i : |a_i - \zeta_i| < 2\}$ for $i \neq j$ and $\{\zeta_j : |a_j - \zeta_j| < 2^n l_3\}$. By the assumption above, $C(2,2j)$ is contained in $M_\epsilon$. Further, by taking the combination $\theta_i = \frac{1}{n}$, we see $(z_1, \ldots, z_n) \in \mathbb{C}^n$ is contained in the logarithmic convex closure of the $C(1,j)$. Let $(b_{j,1}, \ldots, b_{j,n}) \in C(1,j)$ ($j = 1, \ldots, n$) be some points that make this combination.

Since $E_V$ is holomorphic in $\{\zeta_i\}$, in all of $\mathbb{C}^n$, $f$ has a multiple Taylor series; $f(\{s_j\}) = \sum_{m_i \geq 0} c_{m_1 \ldots m_n} s_1^{m_1} \cdots s_n^{m_n}$. This is the Taylor series for $E_V$ in $\{\zeta_i\}$ at $(a_1, \ldots, a_n)$. Using Theorem 10 of Bochner, and the work leading up to it, $E_V$ is holomorphic at $\{z_i\}$, the Taylor series of $E_V$ is at $(a_1, \ldots, a_n)$ effected by the holomorphy of $E_V$ in each $C(1,j)$. We need a bound for $E_V$ in terms of $\{z_i\}$, and so
we will have to incorporate all the coefficients. This really is just a technical point, and we offer a crude solution. Now, in the \( \{ s_j \} \) coordinates, the point \( (a_1, \ldots, a_n) \) has become the origin. Also, in the \( \{ s_j \} \) coordinates, \( C(1, j) \) is translated to the polydisc \( \{ |s_i| < \frac{1}{2} \} \) for \( i \neq j \) and \( \{ |s_j| < 2^n l_3 \} \), and similarly, the constants for the translation of \( C(2, j) \) are 2 for \( i \neq j \) and \( 4^n l_3 \) for \( j \). By abuse of notation, let us call these polydiscs \( C(1, j) \) and \( C(2, j) \) in the \( \{ s_j \} \) coordinates. Now the multi-variable Cauchy formula gives

\[
c_{m_1 \ldots m_n} = \frac{1}{(2\pi i)^n} \int_{\partial C(2, 2j)} f(\{ \zeta_i \}) \zeta_1^{m_1+1} \cdots \zeta_n^{m_n+1} d\zeta_1 \cdots d\zeta_n.
\]

Here \( \partial C(2, 2j) \) denotes the obvious polycircle. Since \( C(2, 2j) \) is contained in \( M_\epsilon \) (in the \( \{ s_i \} \) coordinates), using the estimate (2.1), we see

\[
|f(\{ \zeta_i \})| \ll \epsilon \exp(l (2(n-1) + 4^n l_3 + l_1)\beta).
\]

It follows that

\[
|c_{m_1 \ldots m_n}| \ll \epsilon \frac{1}{2^\left(\sum_{i=1}^n m_i \right) \left(4^n l_3 \right)^m} \exp(l (2(n-1) + 4^n l_3 + l_1)\beta). \tag{2.2}
\]

Let us now estimate \( f(\{ s_i \}) \) using its Taylor series, for \( \{ s_i \} \in C(1, j) \), using absolute values everywhere (for each \( s_i \) and each coefficient). Using (2.2), and the properties that \( C(1, j) \) forces on each \( s_i \), we see

\[
|c_{m_1 \ldots m_n} s_1^{m_1} \cdots s_n^{m_n}| \ll \epsilon \frac{1}{2^\left(\sum_{i=1}^n m_i \right) \left(2^n \right)^m} \exp(l (2(n-1) + 4^n l_3 + l_1)\beta).
\]

Now, the number of ways to write \( m = \sum m_i \) where each \( m_i \geq 0 \) is clearly bounded by \( (m+1)^n \). Since \( n \), the rank of our group is \( \geq 1 \), we see that estimating \( f \) this way, we can take for an upper bound \( \sum_{m \geq 0} \frac{(m+1)^n}{2^m} \) times the above exponential term,
the bound still depending on \( \epsilon \). Since this series converges, we see we can take in estimating \( f \) this way just the above exponential term.

Now, for \((z_1, \ldots, z_n) \in \mathbb{C}^n\), we can take \((b_{j,1}, \ldots, b_{j,n}) \in C(1, j)\) (in the \(\{s_i\}\) coordinates, for each \(j \in \{1, \ldots, n\}\)) so that \((\log(z_1 - a_1), \ldots, \log(z_n - a_n))\) is a combination of the points \((\log b_{j,1}, \ldots, \log b_{j,n})\) \((\theta_i = 1/n)\). Since we are dealing with strips, and will be taking absolute values of exponentials, any ambiguity in a log term does not matter. Further, since \(e^x\) is convex for \(x\) real including \(-\infty\) if any number \(z_i\) or \(b_{i,n}\) is zero, this does not cause any problems in the analysis. Let us write \(F(s) = f(e^s)\). Then the Taylor series for \(f\) gives an exponential series for \(F\) with the same coefficients. Further, \(f((z_i - a_i)) = F(\log(z_1 - a_1), \ldots, \log(z_n - a_n))\), which of course is \(F(\sum \theta_i(\log b_{i,1}, \ldots, \log b_{i,n}))\). Using the convexity of \(e^x\) for \(x\) real, looking for an upper bound, we can separate \(F(\sum \theta_i(\log b_{i,1}, \ldots, \log b_{i,n}))\) into \(\sum \tilde{F}(\Re \log b_{i,1}, \ldots, \Re \log b_{i,n})\), where \(\tilde{F}\) denotes we have taken the absolute values of the coefficients. Now, \(\exp(\Re \log b_{j,1}, \ldots, \Re \log b_{j,n}) \in C(1, j)\), and looking at the coefficients separately is exactly the above estimate for \(f\).

So we have proved for \((z_1, \ldots, z_n)\) not in \(M_{\epsilon}\), we have

\[
|E_V(e, D, \chi, \{z_j\}; \zeta)| \ll \epsilon^\beta \exp(l (2(n - 1) + 4^n l_3 + l_1)^3).
\]

Recall that \(l_3 = (l_1 + l_2 + 1)^n\), where \(l_1\) was the length of \((a_1, \ldots, a_n)\), and \(l_2\) was the length of \((z_1, \ldots, z_n)\). (Of course, both lengths in \(\{s_i\}\) coordinates.) Recall also that \(\beta\) was any number larger than one. It follows easily that if \(\gamma\) is any number larger than \(n\), this estimate combined with estimate (2.1) gives

**Theorem 1** Given \(\epsilon > 0\) and \(\gamma > n\), for \(\{z_i\} \in \mathbb{C}^n\)

\[
|E_V(e, D, \chi, \{z_j\}; \zeta)| \ll \epsilon^\gamma \exp(l(|\{z_j\}|)^\gamma).
\]
Later, we will only be concerned with one variable, i.e., a complex line in $\mathbb{C}^n$. In this situation, the finite order of the resulting entire function can not be more than $n$.

## 2.2 Notes

The above analysis is extremely crude. The radii in the polydiscs can be as much as $|\{z_k\}|^n$. That is why there is an $n$ in the bound. We fully expect to get that $E_V$ is of order one.

Below is (the real projection) a picture for this analysis in rank 2. (Axes, which are not drawn, are perpendicular.)

![Diagram](image.png)

We know the complement of $M_\epsilon$ in $\mathbb{R}^2$ is contained in a small circle. Let $z \in \mathbb{C}^2$, whose real points are in this circle. Now $z \in \mathbb{C}^2$ will be in the logarithmic convex closure of $C(1, 1)$ and $C(1, 2)$. $(a_1, a_2)$ is the center of the intersecting rectangles. Further, $C(2, 1)$ and $C(2, 2)$ (not drawn) are twice as big as $C(1, 1)$ and $C(1, 2)$, with the same center, and have no intersection with the circle. (This was the property of $(a_1, a_2)$.) The bound at $z$ for each coefficient then comes from the bounds in these rectangles, by convexity.
Chapter 3

The Selberg transform

Selberg’s Fredholm determinant proof of the continuation of Eisenstein series involves first constructing a compact operator. This operator is simply a cut-off convolution with a compactly supported point-pair invariant which is usually taken to be smooth. From the spectral properties of the Eisenstein series, one knows how the operator acts when applied to the Eisenstein series, at least in the region of absolute convergence. From taking the resolvent of these operators, one can recover the Eisenstein series; this gives continuation to a larger region. What enters into the resolvent of these operators is the Selberg transform. Our strategy is to control the Selberg transform of a point-pair invariant, so these resolvents remain meromorphic with effective bounds in a vertical strip. For us to do this, if we are using a specific transform, we have to know it is coming from a compactly supported point-pair invariant to be able to use the effective estimates from Fredholm determinants which we will derive later. Since we will be assuming trivial $K$-type, the Selberg transform fits inside the spherical transform in a way we will make precise later. The result of this section, is if we have a holomorphic function on $\mathbb{C}^n$, invariant under the Weyl group, satisfies a Paley-Weiner bound, and otherwise does not decay too rapidly, it is the spherical transform of a compactly supported point-pair invariant which is not
smooth.

We will be using the recent book of Helgason [12] for the theory of the spherical transform. In this section, we assume $G$ is a split, connected, semi-simple Lie group with finite center, and maximal compact subgroup $K$. In later chapters, we will restrict our groups to be real, but the result of this section is valid if $G$ is real or complex. Let $D^b(G)$ denote the space of smooth, compactly supported functions of $G$ which are bi-invariant under $K$. Let $H^R_W(a^*_C)$ denote the set of holomorphic functions $f$ on $a^*_C$ which are invariant under the Weyl group and which satisfy $|f(\lambda)| \leq \frac{C_N}{(1+|\lambda|)^N} e^{R|\lambda|}$ for each natural number $N$. In this situation, $R$ is the Paley-Weiner bound.

Let $\phi_\lambda(g) = \int_K e^{i(\lambda + \rho)(kg)} dk$. Here, we write $G = UAK$, and $A(g)$ is the element of the lie algebra of $A$ whose exponential is the split torus part of this Iwasawa decomposition of $G$. $\phi_\lambda$ is the well known spherical function. It is bi-invariant under $K$, and invariant under the Weyl group in $\lambda$. It is an eigenfunction for $D_K(G)$, the elements of the enveloping algebra which are invariant under $Ad(K)$ (and so in particular, the center).

The spherical transform of $f \in D^b(G)$ is $\tilde{f}(\lambda) = \int_G f(g) \phi_{-\lambda}(g) dg$. Notice that this is the same as the convolution $f * \phi_\lambda(e)$, since $\phi_\lambda(g^{-1}) = \phi_{-\lambda}(g)$. It is known (Theorem 7.1, page 450 of Helgason [12]) that $f \mapsto \tilde{f}$ is a bijection of $D^b(G)$ with $\bigcup_{R>0} H^R_W(a^*_C)$. More specifically, $\text{Supp}(f) \subset B_R(0) \iff f \in H^R_W(a^*_C)$. Here $B_R(0)$ denotes the ball of radius $R$ around the identity element in the symmetric space $G/K$. Obviously, we can identify a function on $G$ as a function of $G/K$, if it invariant under $K$.

The theory of the spherical transform strongly parallels the theory of the Fourier transform. In particular, it is invertible, and an isometry. Before we start quoting theorems, let us mention that central to the theory of the spherical function, appears $c(\lambda)$, the Harish-Chandra $c$-function. We refer the reader to Helgason [12] for more
information about $c(\lambda)$. Now, quoting Theorem 7.5 page 454 of Helgason [12], we have
the Inversion Formula and Plancherel Theorem: for $\tilde{f}(\lambda) \in H^R_W(a_\mathbb{C}_W)$ the spherical
transform is inverted by

$$f(g) = \int_{(a_\mathbb{R})^*} \tilde{f}(\lambda)\phi_\lambda(g)|c(\lambda)|^{-2}d\lambda.$$ 

Further, it is an isometry;

$$\int_G |f(g)|^2dg = \int_{(a_\mathbb{R})^*} |\tilde{f}(\lambda)|^2|c(\lambda)|^{-2}d\lambda$$

and the image of $D^2(G)$ is dense in $L^2(a_\mathbb{R}_W/W, |c(\lambda)|^{-2}d\lambda)$. Here, both equations are
up to a constant that depends only upon the choice of Haar measures $dg$ and $d\lambda$.

Both statements are proved using an asymptotic, analytic expansion of $\phi_\lambda$, in which $c(\lambda)$ appears. All we will need is that $|c(\lambda)|^{-2} \ll 1 + |\lambda|^p$ for $\lambda$ real, and $p = \dim(U)$.

Let us suppose that we are given an entire function $\tilde{f}(\lambda)$ that satisfies the above
Paley-Weiner bound for some $R$, but decays only to $1/(1+|\lambda|)^N$ for some large $N$ and not
$N+1$. We assume $N$ is large enough so that $\tilde{f} \in L^1 \cap L^2(a_\mathbb{R}_W/W, |c(\lambda)|^{-2}d\lambda)$. Let us
define $f(g)$ by the above inversion formula for $\tilde{f}$. Since $\phi_\lambda$ is bounded as a function
of $g$ for $\lambda$ real, it follows that $f$ is defined pointwise. In the remainder of this section,
we will prove that $f$ is compactly supported, has a few derivatives, and has spherical
transform $\tilde{f}$, given $N$ is sufficiently large.

Let $\psi$ be a nonnegative, smooth, bi-invariant function on $G$ which is of compact
support. If the support is small enough, and if $\int_G \psi dg = 1$, it is easy to see that $\tilde{\psi}(0) \neq 0$. By renormalizing, we may assume that we have a function $\psi$ which is
smooth, bi-invariant, $\text{Supp}(\psi) \subset B_1(0)$ and $\tilde{\psi}(0) = 1$. Let us define a sequence of
functions by $\tilde{\psi}_r(\lambda) = \tilde{\psi}(\frac{\lambda}{r})$ for each positive integer $r$. Then each $\tilde{\psi}_r(\lambda)$ is rapidly
decreasing, and satisfies a decreasing Paley-Weiner bound. It follows that the product
$\tilde{f} \cdot \tilde{\psi}_r \in H^{R+1}_W(a_\mathbb{C}_W)$. Let $f_r$ be the inverse transform of $\tilde{f} \cdot \tilde{\psi}_r$. (Formally this is the
convolution of $f$ with the inverse transform of $\tilde{\psi}_r$.)

It follows from the Mean Value Theorem in one variable that $|1 - \tilde{\psi}_r(\lambda)| \leq C/r$ for $|\lambda| \leq r$ where the constant $C$ depends only on $\tilde{\psi}$. Thus, for $d\mu = |c(\lambda)|^{-2}d\lambda$, the Plancherel measure ($\lambda$ real),

$$\int_{a_k^*} |\tilde{f} - \tilde{f} \cdot \tilde{\psi}_r|d\mu \leq \int_{|\lambda| \leq r} |\tilde{f}||1 - \tilde{\psi}_r(\lambda)|d\mu + C \int_{|\lambda| > r} |\tilde{f}|d\mu,$$

for some other constant $C$. This constant comes from the fact that each $\tilde{\psi}_r$ is bounded by the same constant for $\lambda$ real. It follows that

$$\|\tilde{f} - \tilde{f} \cdot \tilde{\psi}_r\|_{L^1(d\mu)} \ll \frac{1}{r} \|\tilde{f}\|_{L^1(d\mu)} + \int_{|\lambda| > r} |\tilde{f}|d\mu.$$

A very similar statement for $L^2$ also holds. So, we see that $\tilde{f} \cdot \tilde{\psi}_r \to \tilde{f}$ in $L^1$ and $L^2$ of $d\mu$.

The inversion formula says $f_r(g) = \int_{\mathbb{R}^n} \tilde{f}(\lambda)\tilde{\psi}_r(\lambda)\phi_{\lambda}(g)|c(\lambda)|^{-2}d\lambda$. Since $\tilde{f} \cdot \tilde{\psi}_r \to \tilde{f}$ in $L^1$, and since $\phi_{\lambda}(g)$ is bounded for $\lambda$ real, it follows that $f_r \to f$ pointwise on $G$. Since $\tilde{f} \cdot \tilde{\psi}_r$ is a Cauchy sequence in $L^2(d\mu)$, by the Plancherel Theorem, so is $f_r$ in $L^2(G)$. Since $f$ is the pointwise limit, we have $f_r \to f$ in $L^2(G)$. Since each $\tilde{f} \cdot \tilde{\psi}_r$ satisfied a Paley-Weiner bound of $R + 1$, each $f_r$ and consequently $f$ is compactly supported. By Cauchy-Schwarz, we have $f_r \to f$ in $L^1(G)$. Let $\hat{f}$ denote the actual spherical transform of $f$; i.e., $\hat{f}(\lambda) = \int_G \tilde{f}(g)\phi_{-\lambda}(g)dg$. By the same formula with $f_r$ and $\tilde{f} \cdot \tilde{\psi}_r$, since $f_r \to f$ in $L^1(G)$, and $\phi_{-\lambda}(g)$ is bounded for $\lambda$ real, it follows that $\tilde{f} \cdot \tilde{\psi}_r \to \hat{f}$ pointwise in $\lambda$ for real $\lambda$. However, by construction, $\tilde{\psi}_r \to 1$ pointwise, and so $\tilde{f} = \hat{f}$ for real $\lambda$. Since they are both holomorphic, we see that what we started with, $\tilde{f}$ really is the spherical transform of $f$. We have determined that $f$ has compact support.

Further, if convergence is fast enough, since $f(g) = \int_{\mathbb{R}^n} \tilde{f}(\lambda)\phi_{\lambda}(g)|c(\lambda)|^{-2}d\lambda$, differentiating $f$ amounts to differentiating the spherical function $\phi_{\lambda}(g)$. Further,
we need only be concerned with derivatives of $\phi_\lambda(g)$ in a compact set, since $f$ is compactly supported. By the above integral representation (over $K$) of $\phi_\lambda(g)$, for $\lambda$ real, it follows that $(D\phi_\lambda)(g)$ is bounded by some polynomial in $|\lambda|$ (depending on $D$) uniformly for $g$ in a compact set for any left-invariant differential operator $D$. (Helgason [12] states this, page 451.) Thus, if we require $f$ to have a certain number of derivatives (number meaning the degree of the differential operator in the enveloping algebra), we just need to increase $N$ sufficiently.

Let us record the result of this density argument:

**Proposition 1** Suppose $\tilde{f}(\lambda)$ is holomorphic in all of $\mathbb{C}^n$, is invariant under $W$, satisfies a Paley-Weiner bound, and otherwise decays polynomially. So $|\tilde{f}(\lambda)| \leq \frac{C_N}{(1+|\lambda|)^N} e^{R|\Im \lambda|}$ for some $N$ and not $N+1$. If $N$ is sufficiently large, then $\tilde{f}$ is the spherical transform of a point-pair invariant $f$ on $G$. $f$ is compactly supported and has a few derivatives. Increasing $N$ increases the number of derivatives $f$ will have.
Chapter 4

Compact Operators and Fredholm Determinants

This chapter is a small preparatory chapter needed for the continuation. From our strategy, we will be selecting a holomorphic function in a strip with specific properties. Using spherical inversion, we will show it is the Selberg transform of a compactly supported function which is not smooth. This chapter shows that if we start with a compactly supported point-pair invariant which is not smooth, then the relevant cut-off integral operators are compact. We obtain a pointwise estimate for the kernel of these integral operators, from which compactness follows. Further, we also obtain a finite order bound for the denominator of the resolvent of these operators.

In this chapter, we will be using the estimates of Godement [8] [9]. For this reason, in this chapter, we assume $G$ is a real, split, connected and reductive linear algebraic group defined over $\mathbb{Q}$. Further, we assume $\Gamma$ is a discrete arithmetic subgroup of $G$ so that $\Gamma \backslash G$ is not compact, yet has finite volume. From this point on, we assume $\Gamma \backslash G$ has one cusp. We will return to the general case in a later section. Let us suppose $\alpha$ is a compactly supported point-pair invariant (bi-$K$-invariant) that is not smooth. If $\alpha$ has enough derivatives, we will show that the kernel of the cut-off integral equation
(operator) is bounded as a function (on $\Gamma \backslash G \times \Gamma \backslash G$). From this, we will be able to get effective bounds on Fredholm determinants, found in the resolvent of this operator.

Let us take $f \in L^1 \cap L^2(\Gamma \backslash G)$. Suppose $P_j$ is the $j^{th}$ standard maximal parabolic with Langlands decomposition $N_jA_jM_j$. For short, let us denote $N_j \cap \Gamma \backslash N_j$ by $N_j^j$. Put $f^j(g) = \int_{N_j^j} f(ng)dn$, the constant term of $f$ along $N_j$. Let us consider $(\alpha * f - \alpha * f^j)(g)$, where * denotes convolution on $G$. By unwinding the definition of $f^j$, and then folding, we see this is an integral operator on $\Gamma \backslash G$;

$$\alpha * f - \alpha * f^j (g) = \int_{\Gamma \backslash G} \left\{ \sum_{\Gamma} \left\{ \alpha(h^{-1}\gamma g) - \int_{N_j^j} \alpha(h^{-1}\gamma ng)dn \right\} \right\} f(h)dh. \quad (4.1)$$

Folding from $N_j \cap \Gamma \backslash G$ instead of $\Gamma \backslash G$ we see the above expression can also be written as

$$\int_{N_j \cap \Gamma \backslash G} \left\{ \sum_{N_j \cap \Gamma} \alpha(h^{-1}\gamma g) - \int_{N_j} \alpha(h^{-1}ng)dn \right\} f(h)dh. \quad (4.2)$$

Following Godement’s two papers [8] [9], we will be making an estimate for the above kernel, for $g$ in a Siegel domain.

Unfortunately, at the time of writing of these papers, the actions of $G$ and $\Gamma$ on $G$ were backwards of what they are today. For brevity of analysis, using Godement’s estimates, we will parallel these papers, but things will look backwards. As usual, let $B$ be a Borel subgroup (minimal parabolic). Let $U$ be its unipotent radical, and $S$ its split torus. We will choose in the center of $S$ in $B$ a closed subgroup $M$ so that the product with $S$ is the center, and $S \cap M$ is finite. (See Godement [8]. This notation is somewhat unfortunate; $M$ has nothing to do with a Levi.) $S^+$ is the connected component of $S$. Let $S^+(t)$ denote the subset of $S^+$ defined by $\alpha_i(s) > t$ for each independent positive root $\alpha_i$, for $t > 0$. If $H$ is any subgroup of $G$, let us denote by $\Omega_H$ a compact set of $H$. We will denote by $\mathcal{S} = \Omega_U \Omega_M S^+(t)K$ a fixed Siegel domain.
Let us consider the above kernel $\sum_{N_j \cap \Gamma} \alpha(h^{-1}g) - \int_{N_j} \alpha(h^{-1}ng)dn$ for $g \in \mathcal{S}$.

Clearly, $N_j \cap \Gamma$ is discrete inside $N_j$. Let $n$ denote the lie algebra of $N_j$. We assume that $N_j \cap \Gamma$ is the image under exp of a discrete subgroup in $n$. (If this is not the case, as Godement [8] points out we could go to a subgroup of finite index in $N_j \cap \Gamma$. This would change things only slightly in our kernel, and in the above folding.) Since exp transfers Haar measures, we can view our kernel as

$$\sum_{n_{\Gamma}} \alpha(h^{-1}\exp(\gamma)g) - \int_{n} \alpha(h^{-1}\exp(n)g)dn,$$

where $n_{\Gamma}$ denotes the full rank lattice in $n$. We haven’t changed much, but the idea is that since $n$ is additive (abelian), this can be shown to be small, by Poisson sum.

If $g \in \mathcal{S}$, we can first ask for what $h$ is $\alpha(h^{-1}ng) \neq 0$, for $n \in N_j$. Since $\mathcal{S} = \Omega_G \Omega_M S^+(t)K$, if we write $g = u_g m_g s_g k_g$ in this decomposition we see $s_g^{-1}g \in \Omega_G$, a compact set of $G$. Thus, $g \in s_g \Omega_G$. Then, since $\alpha$ is compactly supported,

$$\alpha(h^{-1}ng) \neq 0 \Rightarrow h^{-1}ng \in \Omega_G \Rightarrow h \in nG \Omega_G \Rightarrow h \in n(s_g \Omega_G)\Omega_G.$$

So, $h \in ns_g \Omega_G$. (Once again each $\Omega_G$ denotes some compact set, not necessarily the same one each time.) However, since we are going to be integrating in $h$ for $h \in N_j \cap \Gamma\backslash G$, we see that we can take $h \in \Omega_{N_j}s_g \Omega_G$. It follows that $h^{-1}s_g \in \Omega_g s_g^{-1}\Omega_{N_j}s_g \subset \Omega_G$ by the assumptions of $s_g$ and $\mathcal{S}$.

Now, by Poisson sum,

$$\sum_{n_{\Gamma}} \alpha(h^{-1}\exp(\gamma)g) - \int_{n} \alpha(h^{-1}\exp(n)g)dn = \sum_{\tau \neq 0} \int_{n} \alpha(h^{-1}\exp(n)g)e^{2\pi i \tau(n)}dn.$$

Here, the sum is over all nontrivial characters $\tau$ of $n$ which are trivial on $n_{\Gamma}$. Writing
\[ \omega_g = s_g^{-1}g \] and \( \omega_{h,g} = h^{-1}s_g \), we see

\[ \alpha(h^{-1}\exp(\gamma)g) = \alpha(\omega_{h,g}\exp(Ad_{N_j}(s_g^{-1})\gamma)\omega_g). \]

It follows that an individual coefficient

\[ \int_n \alpha(h^{-1}\exp(n)g)e^{2\pi i\tau(n)}dn = \beta_j(s_g) \int_n \alpha(\omega_{h,g}\exp(\gamma)\omega_g)e^{2\pi i\tau(Ad_{N_j}(s_g)n)}dn. \]

Here, \( \beta_j(s) = \det_n Ad_{N_j}(s) \) (recall \( n \) is the lie algebra of \( N_j \)). Let us define

\[ \hat{\alpha}_{h,g}(\tau) = \int_n \alpha(\omega_{h,g}\exp(\gamma)\omega_g)e^{2\pi i\tau(n)}dn. \]

It follows that an individual coefficient is equal to \( \beta_j(s_g)\hat{\alpha}_{h,g}(Ad_{N_j}(s_g)^{-1}\tau) \). Here \( Ad_{N_j}(s_g) \) is the contragredient of the adjoint representation (remember, \( \tau \in n_{\mathbb{R}}^* \)).

Since \( \alpha \) is compactly supported, and since \( \omega_{h,g} \) and \( \omega_g \) remain in compact sets, it follows from integration by parts that

\[ |\hat{\alpha}_{h,g}(\tau)| \leq C_N \|\tau\|^{-N}. \]

Here \( \|\cdot\| \) is a norm on the dual of \( n \). Obviously we are assuming \( \alpha \) has enough derivatives for this particular \( N \). Let us assume \( \alpha \) has sufficient derivatives so that we can take \( N \geq \dim(U) + 2 \). We thus have an upper bound

\[ \int_n \alpha(h^{-1}\exp(n)g)e^{2\pi i\tau(n)}dn \ll \beta_j(s_g)\|Ad_{N_j}(s_g)^{-1}\tau\|^{-N}, \]

where the constant depends on the support and the number of derivatives of \( \alpha \), as well as the particular Siegel domain \( S \).

In this situation, \( N_j \) is minimal; i.e., in other words, if \( \xi \) is a positive root of \( N_j \), then in its decomposition \( \xi = \sum c_i \alpha_i \), we have \( c_j \geq 1 \). It follows that
\| \text{Ad}_{N_j}(s_g)^{-1}\tau \|^{-N} \ll \alpha_j(s_g)^{-N} \| \tau \|^{-N}. \] Here, the constant depends on the Siegel domain. We thus see that the kernel in equation (4.3)

\[ \ll \frac{\beta_j(s_g)}{\alpha_j(s_g)^N} \sum_{\tau \neq 0} \| \tau \|^{-N} \ll \frac{\beta_j(s_g)}{\alpha_j(s_g)^N}. \]

We must now go back to the integral of equation (4.2). We found earlier, that since we are integrating for \( h \in N_j \cap \Gamma \backslash G \), that we could take \( h \in \Omega_{N_j}s_g\Omega_G \), for some compact set \( \Omega_G \). Since the \( UMS+K \) decomposition above is topological, we can write \( \Omega_G = \Omega_U\Omega_M\Omega_{S^+}K \). Then \( h \in \Omega_{N_j}\{s_g\Omega_Us_g^{-1}\}\Omega_M\{s_g\Omega_{S^+}\}K \). Here, \( s_g\Omega_{S^+} \subset S^+(t') \) for some \( t' \). For \( s_g \) fixed, \( s_g\Omega_Us_g^{-1} \) is another compact set in \( U \). It follows that the product above is another Siegel domain, say \( S' \). Actually, we see \( \Omega_{N_j}\{s_g\Omega_Us_g^{-1}\} = s_g\{s_g^{-1}\Omega_{N_j}s_g\}\Omega_Us_g^{-1} \). Since \( s_g^{-1}\Omega_{N_j}s_g \) is a compact set, we see the \( U \) component of \( S' \) can be written as \( s_g\Omega_{U^c}s_g^{-1} \). Let \( S'' \) be the Siegel domain whose \( U \) component is \( U \cap \Gamma \backslash U \) and other components are the same as \( S' \). Then, asymptotically, \( S' \) can be covered by \( \ll C\beta(s_g) U \cap \Gamma \) translates of \( S'' \). Here, \( C = \text{vol}(\Omega_U) \) for the very last \( \Omega_U \), so it depends on the support of \( \alpha \) as well as \( S' \). \( \beta(s) = \det_u \text{Ad}(s) = \prod \xi(s) \), the product being over all the positive roots \( \xi \) of \( U \).

So, suppose \( f \in L^1 \cap L^2(\Gamma \backslash G) \). We see that the integral of equation (4.2) is

\[ \ll \frac{\beta_j(s_g)}{\alpha_j(s_g)^N} \int_{S'} |f(h)| \, dh \ll \frac{\beta_j(s_g)\beta(s_g)}{\alpha_j(s_g)^N} \int_{S''} |f(h)| \, dh \]

\[ \ll \frac{\beta_j(s_g)\beta(s_g)}{\alpha_j(s_g)^N} \| f \|_{L^1(\Gamma \backslash G)} \]  

(4.4)

We recall Wong’s partition of unity (see Wong [35] pages 88-92). In the Iwasawa decomposition \( UAK \), if \( g = nak \), set \( \eta(a) = \max \alpha_i(a) \). Here the maximum is over all positive independent roots. (Let us mention, that we are working with the exponential of a root; a functional on \( A \), not its Lie algebra. We mention this only since it is
different than Wong [35].) Put $a_j = \{a : \alpha_j(a) = \eta(a)\}$. Then, Wong [35] removes a small set which covers where the $a_j$ meet and sets up a partition of unity $\delta_j$ on $A$ so that $\delta_j(a) = 1$ if $a$ is in $a_j$ and not in this small set. Then, we define $\delta_j(g) = \delta_j(a)$ if $g = nak$, and we see $\delta_j(g) = 1$ if $A(g)$ is in most of $a_j$. We will now consider

$$\alpha * f - \sum \delta_j \alpha * f^j (g).$$

First, it is easy to see that $\beta_j(s)\beta(s) \leq \eta(s)^p$ where we can take $p = 2 \dim(U)$. If $g \in S$ is such that $\delta_j(g) = 1$, then the above analysis shows, using (4.4), that

$$\alpha * f - \sum \delta_j \alpha * f^j (g) \ll \eta(s_g)^p N \|f\|_{L^1(\Gamma \setminus G)}.$$

This is true for each $j$ if $\delta_j(g) = 1$. However, with this partition of unity, similar to [35], it is easy to see that the same estimate holds for any $g \in S$, with maybe a larger constant depending only on $N$. In the case of one cusp, we can take a Siegel domain which covers $\Gamma \setminus G$, so we assume the above estimate is valid for any $g \in \Gamma \setminus G$.

Similar to equation (4.1), we see that we can write for $f \in L^1 \cap L^2(\Gamma \setminus G)$

$$\alpha * f - \sum \delta_j \alpha * f^j (g) = \int_{\Gamma \setminus G} k_\alpha(h, g) f(h) dh,$$

for a continuous (in both variables) kernel $k_\alpha$. (Since continuity is a local issue, we can see continuity in both variables by an application of Lemma 9 of Harish-Chandra [10] pages 9 and 10. See also Lemma 2.3.1, page 80 of Wong [35].) By taking $f$ to be in a sequence of an approximate identity that selects out any point $h \in \Gamma \setminus G$, we see that the above estimate gives

$$|k_\alpha(h, g)| \ll \eta(s_g)^{p-N}.$$
In particular, \(k_\alpha\) is bounded on \((\Gamma \setminus G) \times (\Gamma \setminus G)\). Recall that by folding from some \(N_j \cap \Gamma\), we found \(s_h \in s_g \Omega_{S^+}\). So, if \(h\) and \(g\) are in \(\Gamma \setminus G\), \(k_\alpha\) is only supported for \(h\) near \(g\) in terms of the split component of the Siegel domain. In other words, \(k_\alpha\) is only supported if they both go into the cusp at the same rate. Further, if \(N\) is sufficiently large, \(k_\alpha(h, g)\) is *slowly* decreasing. This incorporates the above information into a term usually reserved for a one variable function on a Siegel domain.

We are now in a position to define our operators. It is easy to see with the appearance of the \(\delta_j\) terms, that we have lost automorphy; invariance by \(\Gamma\) on the left. So, we must account for this. Suppose we let \(\mathcal{F}\) be a fundamental domain for \(\Gamma \setminus G\), where the cusp is at infinity. Then for \(f \in L^2(\Gamma \setminus G)\), let us define

\[
K_\alpha f = \left[\alpha * f - \sum \delta_j \alpha * f^j\right]_{\mathcal{F}}
\]

where the bracket denotes *automorphic extension*. (Wong [35] uses this term. Since we will be using and referring to it later, we define it here.) The automorphic extension of any function \(f_1\) on \(G\) is the function \(f_2\) which agrees with \(f_1\) on \(\mathcal{F}\), and is otherwise extended to be automorphic by \(\Gamma\). Then, \(K_\alpha\) is the automorphic extension of an integral operator, where the kernel \(k_\alpha\) is bounded on \(\Gamma \setminus G \times \Gamma \setminus G\). Since \(\Gamma \setminus G\) has finite measure, \(k_\alpha \in L^2(\Gamma \setminus G \times \Gamma \setminus G)\), and \(K_\alpha : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)\). Thus, \(K_\alpha\) is Hilbert-Schmidt, and is easily seen to be compact. Let us record these observations:

**Proposition 2** Let \(\alpha\) be a non-smooth compactly supported point-pair invariant with sufficiently many derivatives. The kernel \(k_\alpha\) of the operator \(K_\alpha\) satisfies \(|k_\alpha(h, g)| \ll \eta(s_g)^{p-N}\), and in particular is bounded pointwise on \(\Gamma \setminus G \times \Gamma \setminus G\). It follows that \(K_\alpha : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)\), is Hilbert-Schmidt and a compact operator.

Suppose we wish to find \(f\) that will solve \((K_\alpha - \lambda I)f = g\) if \(g \in L^2(\Gamma \setminus G)\) and \(\lambda \in \mathbb{C}\). If \(\lambda \neq 0\), then this is the same as \((I - \frac{1}{\lambda} K_\alpha)f = \frac{-1}{\lambda} g = H\), say. If \(\lambda \notin \text{Spec}(K_\alpha)\) this has a unique \(L^2\) solution, say \(f_\lambda\), where we have emphasized the dependence on
λ. From the theory of Fredholm determinants, we can write

$$f_\lambda(h) = \frac{D_\alpha\left(H(h), \frac{1}{\lambda}\right)}{D_\alpha\left(\frac{1}{\lambda}\right)}.$$ 

Here, $D_\alpha$ in the numerator is the limit of a sum of minors of the limit of the determinant defining the denominator (this is Cramer’s rule from linear algebra). This fraction is an explicit form of the resolvent of $K_\alpha - \lambda I$. We see a nonzero $\lambda \notin \text{Spec}(K_\alpha)$ is equivalent to $D_\alpha\left(\frac{1}{\lambda}\right) \neq 0$, and $f_\lambda$ is holomorphic for a given $h$. If $\lambda \in \text{Spec}(K_\alpha)$, and is further nonzero, we see that $f_\lambda$ can pick up poles for a given $h$.

Let us look more closely at the denominator $D_\alpha(s)$ for $s \in \mathbb{C}$. We take Reisz and Nagy [24] as our reference for Fredholm determinants. Instead of partitioning an interval, we must partition $\Gamma \setminus G$, the region of integration into sets of equal measure. As usual, $D_\alpha(s)$ is the limit of the determinant of each partition, the limit taken as these partitions become finer. We can see that in the Taylor expansion of $D_\alpha(s)$, the coefficients are $n$-fold integrals over $\Gamma \setminus G$ of $n \times n$ determinants. If we write $D_\alpha(s) = \sum_n b_n s^n$, we find $b_0 = 1$ and for $n \geq 1$ that $|b_n| \leq V^n M^n \frac{n^{n/2}}{n!}$, where $V$ is the measure of $\Gamma \setminus G$, and $M$ is the bound for $k_\alpha$ on $\Gamma \setminus G \times \Gamma \setminus G$. Here we have used Hadamard’s inequality for determinants (see [24] page 176). By the decay rate of $\frac{n^{n/2}}{n!}$, we see that $D_\alpha$ is entire. Actually, if we compare $\frac{n^{n/2}}{n!}$ for $n$ even to the coefficients of $e^{(2s)^2}$, we see that $\frac{n^{n/2}}{n!}$ decays faster (from the ratio test applied to the sequence of the ratio of the coefficients). See also Levin [21] (page 5). It follows that $D_\alpha(s)$ is of order 2. If we want a uniform estimate, easily, we can find a constant $C$ that depends on $\alpha$ and the measure of $\Gamma \setminus G$ so that $|D_\alpha(s)| \leq C e^{|s|^3}$. (Any number larger that 2 would also suffice. We will treat the numerator in § 5.2.) We have shown:

**Proposition 3** Let $\alpha$ be a point-pair invariant satisfying the assumptions of Proposition 2. Then the Fredholm determinant $D_\alpha(s)$, is an entire function of order two. It follows there is a constant $C$ depending on $\alpha$ so that $|D_\alpha(s)| \leq C e^{|s|^3}$. 

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Chapter 5

Eisenstein series

This chapter is the bulk of this thesis, where we prove our main result, Theorem 2.

In the first section, we define our Eisenstein series, which are constructed from an automorphic cusp form $\Phi$ on the Levi factor of a maximal parabolic subgroup of $G$. Since we will be using spherical inversion, we have to assume $\Phi$ has trivial $K$-type. We list a few simple properties of the Eisenstein series $E(\lambda, \Phi, g)$, and discuss the intertwining operators.

In the second and third sections, we provide effective bounds for the numerators of these resolvent operators for $\lambda$ in a strip, uniformly for $g \in \Gamma \setminus G$. The final estimates of the second section are dependent upon finding a point-pair invariant whose inverse Selberg transform has polynomial growth in the strip. This is done in the third section by inverting the spherical transform. In the third section, we also discuss the resulting continuation of the Eisenstein series for $\lambda$ in the strip. At this point, we can write $E(\lambda, \Phi, g) = \sum J \theta(J, \lambda) F^*_J(\lambda, g)$, where $F^*_J(\lambda, g)$ denotes the (modified) Fredholm solution of an appropriate resolvent, and the $\theta(J, \lambda)$ are matrix coefficients of intertwining operators. We now have effective upper bounds for the numerator and denominator of $F^*_J(\lambda, g)$ for $\lambda$ in the strip, and $g$ in a compact set of $G$. 

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In the fourth section, we discuss how to recover the $\theta(J, \lambda)$ from the $F^\ast_J(\lambda, g)$ evaluated at a finite number of points on the group. We show how this gives each $\theta(J, \lambda)$ is a ratio of two holomorphic functions with finite order estimates for $\lambda$ in the strip. This is what Fredholm theory gives, using our non-smooth kernel method. We want to pull out a finite order Weierstrass product from the denominator to control the poles of the Eisenstein series in the strip. Unfortunately, a finite order bound alone for the denominator is not sufficient to conclude this.

In the fifth section, this problem is solved by appealing to Müller [23], who has shown the $\theta(J, \lambda)$ to be globally meromorphic of finite order. So, to conclude our main result, since $E(\lambda, \Phi, g) = \sum_J \theta(J, \lambda) F^\ast_J(\lambda, g)$, for $\lambda$ in the strip, we need a finite order type of estimate for the number of poles of $F^\ast_J(\lambda, g)$. The theory of Fredholm determinants gives enough analytic information for us to do this. We then use the functional equation in a basic way to conclude the main result of this work, Theorem 2. $E(\lambda, \Phi, g)$ can be written as $F(s, g)/D(s)$, where $D$ and $F$ are entire with computable finite order, and the bounds for $F$ depend smoothly for $g$ in a compact set of $G$. The final two sections are small sections to ensure the main result is valid in the case of several cusps, and in the adelic situation, which we will need to discuss Langlands-Shahidi $L$-functions.

5.1 Preliminaries

In this section, we define the Eisenstein series and list some basic properties. In this chapter, we assume $G$ is a real, split, connected, semi-simple linear algebraic group defined over $\mathbb{Q}$, which is either a classical group or of type $G_2$. Further, we assume $\Gamma$ is a discrete arithmetic subgroup of $G$. These restrictions on $G$ are not necessary for most of the estimates in the second section. However for our particular strategy, we will see in the third section (and in all that follows), we must impose
these conditions. Further, in the first five sections, we assume $\Gamma \backslash G$ has one cusp.

For convenience, if $P_r$ are the standard maximal parabolic subgroups, let us suppose that our Eisenstein series are built from the Levi on $P_1$. More specifically, let $N_1 A_1 M$ be the Langlands decomposition of $P_1$. Let $\Phi \in 0^* L^2(M \cap \Gamma \backslash M, \chi)$ be a cusp form on $M$ with trivial $K$-type. We extend $\Phi$ to a function on $G$ by putting $\Phi(g) = \Phi(m)$, if $g = n a m k$ in this decomposition. Now put $\Phi_\lambda(g) = e^{(\lambda + \rho) \log(a)} \Phi(g)$, again for $g = n a m k$. Here, $\lambda \in a_\mathbb{C}$ and $\rho$ is one half of the sum of the positive roots generating $N_1$. The Eisenstein series is defined by

$$E(\lambda, \Phi, g) = \sum_{\Gamma \cap P_1 \backslash \Gamma} \Phi_\lambda(\gamma g).$$

Let $\alpha$ be the simple root of $A_1$. Set $(a_\mathbb{C}_+)^+ = \{ \lambda \in a_\mathbb{C} : \Re \lambda(H) - \rho(H) > 0 \}$ where $H$ is any element in $a$ so that $\alpha(H) > 0$. It is known that the Eisenstein series converges absolutely and uniformly on compact sets of $(a_\mathbb{C}_+)^+ \times G$. For $\lambda$ in this region of absolute convergence, it is also known that $E(\lambda, \Phi, g)$ is automorphic, is holomorphic for each $g$, has trivial $K$-type, and is an eigenfunction for $Z(G)$, the center of the universal enveloping algebra of $G$. (See Harish-Chandra [10] for basic properties of $E$.)

Suppose $k$ is a compactly supported bi-$K$-invariant function which has sufficiently many derivatives. Under certain restrictions on our group $G$ which we will see later, we have $k \ast \Phi_\lambda(g) = \hat{k}(\lambda) \cdot \Phi_\lambda(g)$. This is Selberg's principle, and $\hat{k}(\lambda)$ is the Selberg/Harish-Chandra (spherical) transform of $k$. $\hat{k}(\lambda)$ only depends on the eigenvalues (of $Z(G)$) and the $K$-type of $\Phi_\lambda$ (which in our case is trivial).

Suppose $P_j$ is a maximal standard parabolic subgroup associate to $P_1$. Let $W(a_1, a_j)$ be the relative Weyl group. It is known that either $j = 1$, and the relative Weyl group has an $SL_2$ structure, or that there is one other associate standard maximal parabolic, and in which case the relative Weyl group has one element. (See
Wong [35] page 142 for details.) We can now present Langlands’ formula. The constant term of $E(\lambda, \Phi, g)$ along $P_j$ is

$$E^j(\lambda, g) = \sum_{s \in W(a_1, a_j)} e^{(s\lambda+\rho_j)\log(a_j)}C(s|\lambda)\Phi(m_j).$$

(5.1)

Here, $a_j$ and $m_j$ are taken from the Langlands decomposition of $P_j$, and $\rho_j$ has its obvious meaning. $C(s|\lambda)$ is an intertwining operator, a linear map

$$C(s|\lambda) : {}^0L^2(M \cap \Gamma\setminus M, \chi) \to {}^0L^2(M_j \cap \Gamma\setminus M_j, ^s\chi),$$

which depends heavily on $\lambda$ (and on $\chi$ which in our case is fixed).

More specifically, say $\{\Phi_p\}_p$ and $\{\Psi_{j,s,n}\}_n$ are bases of $^0L^2(M \cap \Gamma\setminus M, \chi)$ and $^0L^2(M_j \cap \Gamma\setminus M_j, ^s\chi)$ respectively (which are known to be finite dimensional). Then, if $C(s|\lambda)$ has the matrix form $[\theta(j, s, n; p|\lambda)]$ in this basis, we have

$$C(s|\lambda)(\sum c_p\Phi_p) = \sum_n \theta(j, s, n|\lambda)\Psi_{j,s,n}$$

where $\theta(j, s, n|\lambda) = \sum_p c_p\theta(j, s, n; p|\lambda)$. As with the Eisenstein series, $C(s|\lambda)$ is holomorphic and converges uniformly on compact sets in $(\mathfrak{a}_c^\times)^+$. (It is known, of course, from the work of Langlands [20] that each $C(s|\lambda)$ is meromorphic in all of $\mathfrak{a}_c^\times \simeq \mathbb{C}$.)

### 5.2 Back to Fredholm

In this section, we give effective bounds (for $\lambda$ in a strip, and uniformly for $g \in \Gamma\setminus G$) for the numerator of $F_j^{**}(\lambda)(g)$, the solution to an appropriate Fredholm equation. This bound is conditional upon finding a non-smooth point-pair invariant whose inverse Selberg transform has polynomial growth in the strip. Recall, we are assuming our group $G$ is real, split, connected, semi-simple, and defined over $\mathbb{Q}$, which
is either a classical group or of type $G_2$. Further, $\Gamma \backslash G$ has one cusp.

Suppose $P_j$, a standard maximal parabolic, is associate to $P_1$. With the above notation, put

$$I_{j,s,n}(\lambda, g) = \delta_j(g)e^{(s\lambda + \rho_j)\log(a_j)}\Psi_{j,s,n}(m_j).$$

Here, $\delta_j$ is part of the partition of unity we used before. Let $J$ denote the index \{j, s, n\}. Now, let $H_J(\lambda, g) = [I_J(\lambda, g)]_F$. We will be interested in $K_\alpha H_J$ as a function of both $\lambda$ and $g$.

First, let us restrict $\lambda$ to a vertical strip $|\Re \lambda| \leq D_{\rho_1}$ where $D_{\rho_1}$ is large enough so that this strip has an intersection with $(a_c^+)^\perp$ (hence the $\rho_1$). Let us suppose that $S$ is a Siegel domain, as before, and $g \in S$. Then, obviously, any growth of $I_{j,s,n}$ for $g \in S$ comes from the exponential term. We wish to show that $I_{j,s,n}$ is slowly increasing. Thus we need to compare $e^{(s\lambda + \rho_j)\log(a_j)}$ to $\eta(s_g)$. This is easy, since the positive roots of $M_j$ are perpendicular to $A_j$, and since they are also the restriction of the positive roots of $U$, where we have removed any $\alpha_j$. Wong [35] does this (page 110), yet this bound of increase depends on $\lambda$ in a compact set. Since the size of $e^{(s\lambda + \rho_j)\log(a_j)}$ depends on $\Re \lambda$ we see this proof works exactly for $\Re \lambda$ in a compact set; i.e., for $\lambda$ in a vertical strip. We see that $I_{j,s,n}$ is slowly increasing in this strip where the bound depends on $D_{\rho_1}$ and also on $S$. We are more concerned with $H_J$, however, the automorphic extension of $I_{j,s,n}$. Wong [35] proves (pages 112-116) that $H_J$ is slowly increasing. This is a little more difficult, since the Siegel domain enters, and the proof in [35] is limited to $SL_n$. (In a later section, we will apply a lemma of Harish-Chandra [10] exactly for this purpose.) At any rate, citing [35] (or [10] which will be postponed until the future) we have $H_J$ is slowly increasing in this strip, for $g \in S$, where the bound depends on $D_{\rho_1}$ and the Siegel domain.

Let $\alpha$ be a compactly supported point-pair invariant with sufficiently many deriva-
tives. We recall the estimate immediately above equation (4.4). Suppose $g \in S$ so that $\delta_j(g) = 1$. Then

$$\alpha * H_J - \alpha * H_J^j \ll \frac{\beta_j(s_g) \beta(s_g)}{\alpha_j(s_g)} \int_{S''} |H_J(h)| dh.$$  

But $H_J$ is slowly increasing, say $H_J(\lambda, g) \ll \eta(s_g)^d$. This $d$ then depends upon $D_{\rho_1}$ and $S$. Then the same argument as before gives

$$|\alpha * H_J - \alpha * H_J^j| \ll \eta(s_g)^{p+d-N},$$

for $g \in S$ with $p$ and $N$ as above. We see that increasing the number of derivatives $\alpha$ has, we have $K_\alpha H_J(\lambda, g)$ is bounded for $\lambda$ in this strip, uniformly for $g \in \Gamma \setminus G$.

Suppose we wish to solve $(I - sK_\alpha)f_1 = f_2$ for a given $f_2 \in L^2(\Gamma \setminus G)$, and $s \in \mathbb{C}$. Then the above Fredholm fraction is our solution where the denominator is $D_\alpha(s)$. Let us look more closely at the numerator. Let $D_s(g, h)$ denote the $(g, h)$-minor of $D_\alpha(s)$. (See Reisz and Nagy [24], page 174. Here $g, h \in \Gamma \setminus G$.) It is known that each minor has a Taylor expansion. One can see from the same proof of finite order for $D_\alpha(s)$ (Proposition 3), that since $k_\alpha$ is bounded that $|D_\alpha(g, h)| \leq Ce^{s|s|^a}$ for some constant $C$, uniformly for $g, h \in \Gamma \setminus G$. It is known that

$$D_\alpha(f_2(g), s) = f_2(g)D_\alpha(s) + s \int_{\Gamma \setminus G} f_2(x)D_s(g, x) dx. \quad (5.2)$$

Let us go to the Fredholm equation which we wish to solve. We assume $\lambda$ is in a strip for some $D_{\rho_1}$ as above. We wish to find $F_J^{**}(\lambda)$ as a function in $L^2(\Gamma \setminus G)$ so that

$$(K_\alpha - \hat{\alpha}(\lambda))F_J^{**} = -K_\alpha H_J(\lambda).$$
We know the solution is
\[ F_{j}^{**}(\lambda)(g) = \frac{D_\alpha\left(\frac{1}{\alpha(\lambda)}K_\alpha H_J(g), \frac{1}{\alpha(\lambda)}\right)}{D_\alpha\left(\frac{1}{\alpha(\lambda)}\right)}. \]

Suppose we knew \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) in the strip, for some \( Q \). Since \( K_\alpha H_J(g) \) is bounded for \( \lambda \) in the strip and uniformly for \( g \in \Gamma \backslash G \), it follows from equation (5.2) that the numerator of \( F_{j}^{**}(\lambda) \) is \( \ll e^{\lambda^3Q} \) uniformly for \( g \in \Gamma \backslash G \). Further, it is easy to see the derivative with respect to \( \lambda \) of \( H_J \) is slowly increasing for \( \lambda \) in the strip, with almost the same bound as \( H_J \). (Differentiating pulls out a \( \log a_j \) term, which increases slower than any power, in the Siegel domain.) If the decrease of the slowly decreasing kernel \( k_\alpha \) is sufficiently large, it is easy to see all integrals above converge fast enough to conclude \( F_{j}^{**}(\lambda)(g) \) is holomorphic for \( \lambda \) in the strip for a fixed \( g \). Let us record this result:

**Proposition 4** Given \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) for some \( Q \) for \( \lambda \) in a strip, the numerator of \( F_{j}^{**}(\lambda)(g) \) is \( \ll e^{\lambda^3Q} \) uniformly for \( g \in \Gamma \backslash G \). Further, for a fixed \( g \), \( F_{j}^{**} \) is holomorphic for \( \lambda \) in the strip.

### 5.3 A particular point-pair invariant

In this section, for \( \lambda \) in a given strip, we will produce a point-pair invariant \( \alpha \) so that \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) in this region. We will see most of the technical reasons in this section why we impose the restrictions that our group \( G \) is to be real, split over \( \mathbb{Q} \), connected, and semi-simple which is either a classical group or of type \( G_2 \). Recall, we are still assuming \( \Gamma \backslash G \) has one cusp.

Suppose \( \alpha \in C_\mathbb{Q}^c(K \backslash G/K) \) where the \( Q \) denotes that \( \alpha \) is not smooth, but has sufficiently many derivatives. Recall that \( \Phi \) is our cusp form on \( M \), and \( \Phi_\lambda(g) = e^{(\lambda + \rho)\log(a)}\Phi(g) \), if \( g = namk \). Here, \( M \) is the Levi factor of a maximal parabolic
subgroup \( P \) and we can write \( P = NAM \) as usual. We assume \( M \) is generated by a subset of positive independent roots, where we have deleted \( \alpha_1 \). By assumption, \( \Phi \) is an eigenfunction of \( \mathcal{Z}(M) \), and it is well known that \( \Phi_\lambda \) is an eigenfunction of \( \mathcal{Z}(G) \) (for instance, see [10] page 29). Under certain restrictions on our real group, \( G \), (which we will now discuss) we have \( \alpha * \Phi_\lambda(g) = \hat{\alpha}(\lambda) \cdot \Phi_\lambda(g) \), and that \( \hat{\alpha} \) depends only on the eigenvalues (of \( \mathcal{Z}(G) \)) and \( K \)-type of \( \Phi_\lambda \), which in our case is trivial. A possible way to show \( \alpha * \Phi_\lambda(g) = \hat{\alpha}(\lambda) \cdot \Phi_\lambda(g) \) is outlined in Wong [35] (pages 52-72) (of course, this reference is influenced by Selberg [26]). The result would follow using Selberg’s symmetrization operator (cf. Wong [35] pages 61-69) if we knew apriori that \( \Phi_\lambda(g) \) was already an eigenfunction of \( D_K(G) \). Now, there is an obvious surjective homomorphism \( \mu : D_K(G) \to D(G/K) \) with kernel \( D_K(G) \cap D(G)k \) (see Helgason [12]). Here, \( D(G) \) and \( D(G/K) \) are the left invariant differential operators of \( G \) and of the symmetric space \( G/K \), and \( k \) is the Lie algebra of \( K \). It follows that \( D_K(G) = D(G/K) + D_K(G) \cap D(G)k \). The question at hand then becomes when is \( D(G/K) \) the image of \( \mathcal{Z}(G) \) under the map \( \mu \). If this is true, Wong [35] proves that an eigenfunction of \( \mathcal{Z}(G) \) will automatically by an eigenfunction of \( D_K(G) \). This property happens to be true if our real, split group \( G \) is a real classical group, or of type \( G_2 \), for example (cf. Helgason [12] page 326). From this point on, we will restrict our real group \( G \) to be of this form.

Consider \( \Upsilon_\lambda(g) = \int_K \Phi_\lambda(kg)dk \). Here, \textbf{we assume} \( \Phi(e) \neq 0 \). (This is necessary in order to apply the theory of the spherical transform.) Since \( \Phi \) is an eigenfunction for \( \mathcal{Z}(M) \), \( \Phi_\lambda \) will be an eigenfunction for \( \mathcal{Z}(G) \). From the above discussion, with our assumptions on our group, \( \Phi_\lambda \) will thus be an eigenfunction for \( D_K(G) \). Then \( \Upsilon_\lambda \) has the same eigenvalues under \( D_K(G) \) as \( \Phi_\lambda \), and the same \( K \)-type; they both can be viewed as analytic functions on \( G/K \). It is a theorem of Helgason [11] that up to a constant, \( \Upsilon_\lambda(g) \) must be of the form \( \int_K e^{(\gamma(\lambda,\Phi)+\rho_0)A(kg)}dk \), since it is trivial under \( K \) on the left. All we know initially is that the point \( \tau(\lambda,\Phi) \in \mathbb{C}^n \) depends upon \( \lambda \).
and $\Phi$. Thus, $\hat{\alpha}$ is a spherical transform. We need to pin down the dependence of this point on $\lambda$ and $\Phi$.

It is a general fact that if $\phi$ is an eigenfunction of $Z(G)$, which is $K$-finite on the right (which is coming from an admissible representation of $G$) that the actual eigenvalues of $Z(G)$, the infinitesimal character, are determined by a point in the dual of $h$, up to conjugacy by the Weyl group. Here, $h$ is the complexification of a Cartan subalgebra (see Knapp [18] pages 223-226). Let $h_m$ denote the complexification of a Cartan subalgebra of $M$. In the Langlands decomposition $P = NAM$, let $h_1$ be the complexification of the Lie algebra of $A$ (which by assumption is one dimensional). Then $\Phi$ being a cusp form on $M$ corresponds to an irreducible constituent of the right regular representation on $L^2(M \cap \Gamma \backslash M)$. So, there is a point $\tau_m$ in the dual of $h_m$ which determines the eigenvalues of $Z(M)$ on $\Phi$ up to conjugacy by the Weyl group of $M$. It follows from Proposition 8.22 of Knapp [18] (page 225) that the point $\tau(\lambda, \Phi) = \lambda + \tau_m$ relative to the basis $h_1 \oplus h_m$.

We see that $\tau(\lambda, \Phi)$ is just a line (not necessarily unique) in the dual of $h$, a Cartan subalgebra. Recall that $\lambda$ is in the direction of $\rho_1$; i.e., half the sum of positive roots of $N$. This is from the $h_1$ embedding. However, there might be a more natural basis of $h$ coming from the Killing form (on the real Lie algebra of our real group $G$). If $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ are coordinates in this natural basis, we see we have $\lambda_i = \tau_i + r_i \lambda$ for some point $(\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ and real numbers $r_i$. (These $r_i$ just pin down the direction of $\rho_1$ in the natural basis.) It follows that up to a constant our Selberg transform $\hat{\alpha}(\lambda) = \hat{\tilde{\alpha}}(\{\tau_i + r_i \lambda\})$. Here the tilde denotes the spherical transform (as above).

In light of Proposition 1, we can find $\alpha$ by inverting the spherical transform. So, we need to construct $\tilde{f}(\{\lambda\}_i)$ which is invariant under $W$, satisfies a Paley-Weiner bound, and otherwise decays to some polynomial rate. Then, by Proposition 1, there exists $\alpha \in C^0_c(K \backslash G/K)$ so that $\tilde{\alpha}(\lambda_i) = \tilde{f}(\lambda_i)$. Then our Selberg transform in one
variable is \( \hat{\alpha}(\lambda) = \tilde{\alpha}(\{\tau_i + r_i\lambda\}) \).

So, consider

\[
f_{q,c}(\{\lambda\}_j) = \left( \prod_{w \in W} \frac{\sin(i \frac{w(\lambda)}{q} + c)}{i \frac{w(\lambda)}{q} + c} \right)^q,
\]

for \( q \in \mathbb{Z}^+ \) and \( c \in \mathbb{C} \). Here the product is over the Weyl group, and \( i = \sqrt{-1} \).

Clearly, this is invariant under \( W \), of Paley-Weiner type in the real direction, and otherwise of polynomial decay. If \( q \) is sufficiently large, it follows that \( f_{q,c} \) will produce a sufficient \( \alpha \). The \( q \) in \( \frac{\{\lambda\}_j}{q} \) guarantees that the support of the corresponding \( \alpha \) does not depend on \( q \). So, if we need to increase the number of derivatives \( \alpha \) has, we just need to increase \( q \). Let’s determine where the zeros of \( \hat{\alpha}(\lambda) = f_{q,c}(\{\tau_j + r_j\lambda\}) \) can be.

(For the spherical function Helgason [12] uses \( i\{\lambda\}_j \) instead of \( \{\lambda\}_j \). That is why we need a Paley-Weiner bound in the real direction.)

If \( c = 0 \) the zeros are all parallel to the unitary axis, and contained within a fixed distance, since \( \{\tau_j\} \) is fixed. (Of course, assuming the fixed points \( \{\tau_j\}, \{r_j\} \) are such that \( f_{q,c}(\{\tau_j + r_j\lambda\}) \) does not vanish identically. We could change \( c \) slightly to ensure this does not happen.) Suppose all the zeros of \( \hat{\alpha} \) are contained in \(|\Re \lambda| \leq D_1\). By enlarging \( D_1 \) if necessary, we assume \( \Re \lambda \geq \frac{D_1}{5} \) is contained in the region of absolute convergence. Consider \( c = 40i \frac{D_1}{q} \). (We can change this slightly to ensure \( f_{q,c}(\{\tau_j + r_j\lambda\}) \) does not vanish identically.) By enlarging the constant 40 depending on the minimum absolute value of the nonzero terms of \( < \alpha_1, w \cdot \rho_1 > \) for \( w \in W \), this will shift the zeros to the right and the left by at least \( 10D_1 \). This leaves a strip \(|\Re \lambda| \leq 5D_1\) in which \( \alpha \) satisfies \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) for some \( Q \). We can see \( Q \leq q \cdot |W| \).

Let us record this result:

**Proposition 5** Let \( \Re \lambda \geq \frac{D_1}{5} \) be contained in the region of absolute convergence.

Then if \( q \) is sufficiently large, \( f_{q,c} \) produces a compactly supported point-pair invariant \( \alpha \) that satisfies \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) in \(|\Re \lambda| \leq 5D_1\).
We thus have constructed a point pair invariant with the desired property needed in Proposition 4. Combining Propositions 3 and 4, for $\lambda$ in $|\Re \lambda| \leq 5D_1$ (as above) we see $F_J^{**}(\lambda)$, the solution of

$$(K_\alpha - \hat{\alpha}(\lambda))F_J^{**} = -K_\alpha H_J(\lambda)$$

can be written as $F_J^{**}(\lambda)(g) = \frac{A_J^{**}(\lambda, g)}{B_J(\lambda)}$ where $A_J^{**}(\lambda, g) \ll e^{\lambda|Q^3}$, uniformly for $g \in \Gamma\backslash G$, and $B_J(\lambda) = D_\alpha(\hat{\alpha}(\lambda)^{-1}) \ll e^{\lambda|Q^3}$.

Now, let $F_J^*(\lambda) = F_J^{**}(\lambda) + H_J(\lambda)$. $H_J(\lambda, g)$ for a fixed $g$ is easily seen to be of fixed growth in the strip. However $g$ could go into the cusp, which destroys any estimate. So, if $g$ remains in a compact set it is easy to see that $H_J(\lambda)$ retains a trivial finite order estimate, where the constant depends on this compact set. It follows that:

**Proposition 6** For $g \in \Omega_G$, a compact set, and $\lambda$ in a strip we can write $F_J^*(\lambda)(g) = \frac{A_J(\lambda, g)}{B_J(\lambda)}$, where $B_J \ll e^{\lambda|Q^3}$, and $A_J(\lambda, g) \ll \Omega_G e^{\lambda|Q^3}$. Further, $B_J$ is holomorphic in the strip, and so is $A_J(\lambda, g)$ for each fixed $g$.

If we take $q$ large enough, one can see for $\lambda$ in the intersection of the strip and the region of absolute convergence (and off $\text{Spec}(K_\alpha)$), that

$$(K_\alpha - \hat{\alpha}(\lambda))E(\lambda, \Phi, g) = -\hat{\alpha}(\lambda) \sum_J \theta(J|\lambda)H_J(\lambda, g).$$

(See equation (5.1) and the above notation.) If we define

$$F = \sum_J \theta(J|\lambda)F_J^*(\lambda)$$

we see that for $\lambda$ in the above region, $E(\lambda, \Phi, g) = F(\lambda, g)$. However, $F_J^*$ is meromorphic in the strip, and so given the meromorphy of the coefficients $\theta(J|\lambda)$, this gives the meromorphic continuation of the Eisenstein series to the strip. (Wong [35] has shown that there are no connectedness problems arising from $\hat{\alpha}$ and $\text{Spec}(K_\alpha)$.)
Now, however, we have effective bounds for the $F_J^*$. 

### 5.4 The Intertwining Operator

In this section, we will discuss what the Fredholm theory establishes for these intertwining operators $\theta(J|\lambda)$, using our method of non-smooth kernels. We obtain that the $\theta(J|\lambda)$ are meromorphic of finite order in the strip $|\Re \lambda| \leq D_1$. We assume the same restrictions imposed on $G$ and $\Gamma \setminus G$ as the previous section.

In the above sum

$$F = \sum J \theta(J|\lambda)F_J^*(\lambda)$$

we recall the index $J = \{j, s, n\}$ and $\theta(J|\lambda)$. Here $j$ indicates that $P_j$ is a standard maximal parabolic associated to $P_1$. $s \in W(a_1, a_j)$, the relative Weyl group. $n$ is the dimension of $^0L^2(M_j \cap \Gamma \setminus M_j, ^s\chi)$, a basis being $\{\Psi_{j,s,n}\}_n$. For $\{\Phi_p\}_p$ a basis of $^0L^2(M \cap \Gamma \setminus M, \chi)$, if $\Phi = \sum_p c_p \Phi_p$, we have $\theta(j, s, n|\lambda) = \sum_p c_p \vartheta(j, s, n; p|\lambda)$ if $[\vartheta(j, s, n; p|\lambda)]$ is the matrix form of the intertwining operator $C(s|\lambda)$ for these bases.

We do not have to consider other $j$, since the constant term of our Eisenstein series is zero along $P_j$ if $P_j$ is not associate to $P_1$, at least in the region of absolute convergence.

First following Wong [35] (page 135), we will be able to express the $\theta(J|\lambda)$ in terms of expressions involving the $F_J^*(\lambda)$ evaluated at some points on $G$.

Let $J_1$ denote the index $\{1, 1, n\}$. Let $J_2$ be any index of $J$ not of the form $J_1$. In other words, if $J_2 = \{j, s, n\}$, then $s$ must be nontrivial if $j = 1$.

If $D \in \mathcal{Z}(G)$ let us define

$$A_{D,J}(\lambda, g) = DF_J^*(\lambda, g) - \chi_\lambda(D)F_J^*(\lambda, g).$$
Here, $\chi_\lambda(D)$ is the eigenvalue of $D$ on $E$, coming from the eigenvalues $\chi$ of $\Phi$.

If $P_i$ is associate to $P_1$ let us define

$$B_{i,J}(\lambda, g) = \int_{N_i \cap \Gamma \backslash N_i} F_J^*(\lambda, ng) dn - \delta_{ij} e^{(s\lambda + \rho_i) \log a_i} \Psi_{j, s, n}(m_j).$$

Here $\delta_{ij}$ is the Kronecker delta, and $i$ is in the same index set as $j$. (Recall, in the case of one cusp there can be at most one other standard maximal parabolic associate to $P_1$. So, the index set $j$ lies in consists of either one or two elements. We refer the reader back to the first section of this chapter to recall the relative Weyl group information.) Let $P_l$ denote a standard maximal parabolic not associate to $P_1$. Let us define

$$C_{l,J}(\lambda, g) = \int_{N_l \cap \Gamma \backslash N_l} F_J^*(\lambda, ng) dn.$$

Finally, let us put

$$D_{k,J}(\lambda, g) = F_J^*(\lambda, gk) - F_J^*(\lambda, g).$$

Now, we know in the situation $j = 1$, for $s$ the identity element in $W(a_1, a_1)$, the intertwining operator $C(s|\lambda) = I$. It follows that the $\theta(J_1)$ are all constants.

For $\lambda$ in the region of absolute convergence, using that $E(\lambda, \Phi, g)$ is an eigenfunction for $Z(G)$, its constant terms have a specific form (from Langlands’ formula in the case of associate parabolics and zero otherwise), and that $K$ must act trivially on the right, we obtain the following set of equations in the $\theta(J, \lambda)$.

$$\sum_{J_2} \theta(J_2, \lambda) A_{D, J_2}(\lambda, g) = - \sum_{J_1} \theta(J_1, \lambda) A_{D, J_1}(\lambda, g) \quad (5.3)$$

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\[ \sum_{J_2} \theta(J_2, \lambda) B_{i,J_2}(\lambda, g) = - \sum_{J_1} \theta(J_1, \lambda) B_{i,J_1}(\lambda, g) \]  
(5.4)

\[ \sum_{J_2} \theta(J_2, \lambda) C_{l,J_2}(\lambda, g) = - \sum_{J_1} \theta(J_1, \lambda) C_{l,J_1}(\lambda, g) \]  
(5.5)

\[ \sum_{J_2} \theta(J_2, \lambda) D_{k,J_2}(\lambda, g) = - \sum_{J_1} \theta(J_1, \lambda) D_{k,J_1}(\lambda, g) \]  
(5.6)

Equation (5.3) is for \( D \in Z(G) \). Equation (5.4) is for all \( i \) in the same index set as \( j \). Equation (5.5) is for all \( l \) so that \( P_l \) is a standard maximal parabolic not associate to \( P_1 \). Equation (5.6) is for all \( k \in K \).

We recall Proposition 6; each \( F_J^\ast \) can be written as \( F_J^\ast(\lambda)(g) = A_J(\lambda, g) B_J(\lambda) \). For \( \lambda \) in the strip, and for \( g \in \Omega_G \), a compact set we recall that \( A_J(\lambda, g) \ll \Omega_G e^{\lambda |\lambda|^2} \), and \( B_J(\lambda) \ll e^{\lambda |\lambda|^2} \) with no dependence on \( G \). By looking at their definitions, it follows easily that \( B_{i,j}(\lambda, g), C_{l,j}(\lambda, g), \) and \( D_{k,j}(\lambda, g) \) each can be written in the same way, with the same properties. \( A_{D,J}(\lambda, g) \) will take a little bit of justification.

Since \( F_J^\ast = F_J^{**} + H_J \), let us start with \( F_J^{**} \). The numerator has exactly the form of equation (5.2), but we need to replace \( s \) with \( \hat{\alpha}(\lambda)^{-1} \), replace \( f_2 \) with \( K_\alpha H_J(\lambda, g) \) and multiply by \( \hat{\alpha}(\lambda)^{-1} \). It follows, that if we want to differentiate this object with respect to \( D \in Z(G) \), we need bounds on \( D(K_\alpha H_J(\lambda, g)) \) and \( D(D_s(g, x)) \). Here, again, \( D_s(g, h) \) is the \((g, h)\)-minor of \( D_\alpha(s) \).

Now, the operator \( K_\alpha \) has an integral kernel, \( k_\alpha \). If convergence is fast enough, then differentiating \( \int_{\Gamma \setminus G} k_\alpha(h, g) f(h) dh \) should amount to differentiating \( k_\alpha \) in the \( g \) variable. In our case, for \( \lambda \) in the strip, \( H_J(\lambda, g) \) is slowly increasing, so this is justified, as long as \( \alpha \) has enough derivatives. Since \( g \) is on the right in every expression involving \( k_\alpha \) (example, \( \alpha(h^{-1} g) \)), it follows that \( D(k_\alpha(h, g)) = k_{(D_\alpha)}(h, g) \).

So, exactly the same argument as above gives \( D(K_\alpha H_J) \) is bounded in the strip, as
long as $D\alpha$ has sufficient derivatives, which forces even more on $\alpha$. So, we assume in what follows that in equation (5.3) we only need those $D$ which generate $Z(G)$, in particular, only finitely many. This is justified, because more operators than just the generators gives redundant information on a function which is smooth. We cannot conclude that any $F_j^*$ is smooth on $G$ for a fixed $\lambda$. So, initially, we cannot conclude $F = \sum_J \theta(J|\lambda)F_J^*(\lambda)$ is smooth for a fixed $\lambda$. However, the constant functions $\theta(J|\lambda)$ make the above sum an eigenfunction of $Z(G)$ (it was set up this way). One of these operators is the Casimir element, and it follows that $F$ will be smooth by ellipticity.

Now for the second term, the integral in equation (5.2). Since $K_\alpha H_J$ is bounded for $\lambda$ in the strip, we just need to control $D(D_s(g,x))$. By equation (43) of Reisz and Nagy [24] (page 174) in each Taylor coefficient of $(D_s(g,x))$, only one row of the determinant is affected by $g$. So, to get a finite order bound, it is sufficient that $D(k_\alpha(g,x))$ is bounded on $\Gamma \setminus G \times \Gamma \setminus G$ (given that of $k_\alpha$, which we have). However, this is the kernel of $D\alpha$. This is part of Proposition 2 applied to $D\alpha$.

In the definition of $H_J$, a partition of unity $\delta_j$ can be chosen with all of the necessary properties, but also so that its derivatives of all the generators of $Z(G)$ are tame. It follows that for $g \in \Omega_G$, $H_J(\lambda,g)$ will satisfy a trivial finite order bound in the strip, where the constant depends on the compact set $\Omega_G$.

So we have that $A_{D,J}(\lambda,g)$ can be written in the desired fraction form with similar bounds. (We are only taking the generators of $Z(G)$.) From the uniqueness principle (see Wong [35] pages 140-150), it follows that at some point $\lambda$ in this strip we can find a finite number of points $\{x_1, x_2, \ldots\} \subset G$, so that we can write each $\theta(J|\lambda)$ as a ratio of determinants involving the $A_{D,J}(\lambda,g)$, $B_{i,J}(\lambda,g)$, $C_{i,J}(\lambda,g)$, and $D_{k,J}(\lambda,g)$ evaluated at these points $\{x_1, x_2, \ldots\}$. If we fix this expression, and let $\lambda$ vary in the strip this gives a meromorphic continuation to the strip. Let $\Omega_G$ be any compact set which contains $\{x_1, x_2, \ldots\}$. Referring to Proposition 6, we thus have
Proposition 7 For $\lambda$ in such a strip, each $\theta(J,\lambda)$ can be written as a ratio

$$\theta(J,\lambda) = \frac{a_J(\lambda)}{b_J(\lambda)}$$

where $a_J(\lambda), b_J(\lambda) \ll_{\Omega_G} e^{|\lambda|^3Q}$, and both are holomorphic in the strip.

Unfortunately, with only this upper bound for $b_J(\lambda)$ for $\lambda$ in a strip, one cannot conclude that $b_J$ has a polynomial number of zeros (in $R$) up to height $R$.

5.5 The Main Result, using Müller

From the discussion immediately following Proposition 6, we see our Eisenstein series has a meromorphic continuation to the strip. More specifically, for $\lambda$ in the strip,

$$E(\lambda, \Phi, g) = \sum_J \theta(J|\lambda) F_J^*(\lambda)(g). \quad (5.7)$$

Proposition 6 gives effective bounds for $F_J^*(\lambda)(g)$ for $\lambda$ in the strip and $g$ in a compact set. Proposition 7 gives a somewhat effective bound for $\theta(J,\lambda)$ for $\lambda$ in a strip. In light of the above discussion, this is not enough analytic information to pull out a finite order Weierstrass type product containing the poles of $E$ for $\lambda$ in the strip. So, at this point, we refer to Müller [23] who has shown the $\theta(J,\lambda)$ are meromorphic of finite order in all of $\mathbb{C}$. We then have enough analytic information from the form of the denominator of $F_J^*(\lambda)$ to conclude our main result. This part is an application of the analytic properties we have seen of Fredholm determinants. Let us reiterate that the author hopes to remove the dependence of Müller’s estimates to establish a self-contained result. This thesis is a first attempt at this goal.

We recall our restrictions on $G$. $G$ is to be real, split over $\mathbb{Q}$, connected, and semi-simple which is either a classical group or of type $G_2$. We are still assuming
\[ \Gamma \backslash G \] has one cusp.

Now each \( \theta(J, \lambda) \) is a sum of matrix coefficients of the intertwining operator \( C(s \mid \lambda) \). So, by Theorem (5.10) of Müller [23], and the paragraph preceding it we see each \( \theta(J, \lambda) \) can be written as a ratio of entire functions

\[ \theta(J, \lambda) = \frac{a_J(\lambda)}{b_J(\lambda)} \]

where \( a_J(\lambda), b_J(\lambda) \ll e^{\|\lambda\| n_1 + 3} \) for \( \lambda \in \mathbb{C} \). Here, \( n_1 \) is the dimension of the manifold \( G/K \).

Now, by Hadamard’s Theorem, we can assume each denominator, \( b_J \) is only a Weierstrass product. By taking a common denominator, (so absorbing parts of products into the numerators \( a_J \)), we may assume each \( b_J(\lambda) \) is the same Weierstrass product, say \( b(\lambda) \). We thus can assume \( \theta(J, \lambda) \) can be written as \( \frac{a_J(\lambda)}{b(\lambda)} \) with the same estimates.

Let us recall that \( F_J^{**}(\lambda)(g) \) is the solution to

\[ (K_\alpha - \hat{\alpha}(\lambda))F_J^{**} = -K_\alpha H_J(\lambda). \]

It follows that the denominator of \( F_J^{**}(\lambda)(g) \) (and thus the same will be true for \( F_J^*(\lambda)(g) \)) does not depend on \( J \), and of course is independent of \( g \). In the above notation, this denominator is

\[ D_\alpha \left( \frac{1}{\hat{\alpha}(\lambda)} \right). \]

Let us recall Proposition 3, and the discussion leading up to it. If \( s \in \mathbb{C} \), then \( D_\alpha(s) \) is entire, and satisfies \( |D_\alpha(s)| \leq Ce^{\|s\|^3} \) for some constant depending on \( \alpha \). Further, in its Taylor expansion \( D_\alpha(s) = \sum_n b_ns^n \), we have \( b_0 = 1 \). For \( \lambda \) in a strip, let us write
$d(\lambda) = D_\alpha(\frac{1}{\alpha(\lambda)})$. It follows by the Fredholm representation of $E$ in equation (5.7) we can pull out the common denominator $b(\lambda) \cdot d(\lambda)$ from every term in the sum, for $\lambda$ in the strip. We will show that for $\lambda$ in a strip, we can factor out a Weierstrass type product from $d(\lambda)$. Further, we will show the quotient (which is nonvanishing) is not too small (in term of $\lambda$).

Let us assume $\alpha$ satisfies the result of Proposition 5. So, $\hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q$ in the strip $S = \{\lambda : |\Re \lambda| \leq D_1\}$, with the above property for $D_1$. ($\alpha$ satisfies this for $5D_1$, but we will take just $D_1$ in $S$.) We will count how many zeros $d(\lambda)$ can have in $S_R = \{|\Im \lambda| \leq R\} \cap S$. Now, $\hat{\alpha}(\lambda)^{-1}$ maps $S_R$ to a set contained in a circle of radius $\ll R^Q$. Then, the standard estimate for functions of finite order (using Jensen’s formula) applied to $D_\alpha(s)$ gives the number of zeros for $s$ in such a region is $\ll R^3Q$. We must now count how many points $\lambda \in S_R$ are mapped to such zeros $s$ by $\hat{\alpha}(\lambda)^{-1}$. Since $\hat{\alpha}(\lambda)^{-1}$ is asymptotically polynomial, this is easily seen to be finite (for example by a Poisson-Jensen formula centered about height $R$, with radius out to the strip, for example). It follows that the number of zeros $d(\lambda)$ has in $S_R$ is $\ll R^3Q$. Let $d_p(\lambda)$ be the standard Weierstrass product built from the zeros of $d(\lambda)$ in $S$. Then $d_p(\lambda)$ is of finite order $3Q$ in all of $\mathbb{C}$. The product $b(\lambda) \cdot d_p(\lambda)$ will be our global denominator.

Let $\Omega_G$ be a compact set of $G$. We will estimate $b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g)$ for $\lambda \in S$, and $g \in \Omega_G$. By equation (5.7) and our simplifying assumptions above, we need a finite order upper bound for each

$$m_J(\lambda, g) \overset{\text{def}}{=} a_J(\lambda) \frac{NF_J^*(\lambda)(g)}{d(\lambda)} \frac{d(\lambda)}{d_p(\lambda)}$$

for $\lambda \in S$ and $g \in \Omega_G$. Here $NF_J^*$ denotes the numerator of $F_J^*(\lambda)(g)$. Now, if $g \in \Omega_G$, this expression is holomorphic for $\lambda \in S$. Let $\lambda_0 = iR$, and define $C_R$ to be the disc $\{\lambda : |\lambda - \lambda_0| \leq D_1\}$. Let $C'_R$ be the disc $\{\lambda : |\lambda - \lambda_0| \leq 2D_1\}$. So $C_R$ just
meets the boundary of $S$. Further, all relevant estimates are still valid for $\lambda \in C'_R$, by the construction of $D_1$. By the Poisson-Jensen formula, it follows that
\[
\log^+ |m_J(\lambda, g)| \leq \frac{1}{2\pi} \int_{\partial C'_R} \log^+ |m_J(s, g)| P_{C'_R}(s, \lambda) d\theta_s,
\]
for $\lambda \in C_R$. Here, $s = iR + 2D_1 \exp(i\theta_s)$, and $P_{C'_R}$ is the obvious Poisson kernel. Obviously, $P_{C'_R} \leq C''$, depending only on the constants (coefficients of $D_1$) defining $C_R$ and $C'_R$, not on $D_1$ or $R$. We see (dropping $d\theta_s$ for convenience)
\[
\log^+ |m_J(\lambda, g)| \ll \int_{\partial C'_R} \log^+ |a_J(s)| + \int_{\partial C'_R} \log^+ |NF_\gamma(\lambda)(g)| + \int_{\partial C'_R} \log^- \left| \frac{d(\lambda)}{d_p(\lambda)} \right|.
\]
Using Proposition 6, the second term on the right above is $\ll_{\Omega_c} R^{1|\alpha|Q}$. Müller [23] gives (as explained above) the first term is $\ll R^{1|\alpha|n+3}$. So, we need to estimate the last term.

Now, we always have
\[
\int_{\partial C'_R} \log^- \left| \frac{d(\lambda)}{d_p(\lambda)} \right| = \int_{\partial C'_R} \log^+ \left| \frac{d(\lambda)}{d_p(\lambda)} \right| - \int_{\partial C'_R} \log \left| \frac{d(\lambda)}{d_p(\lambda)} \right|.
\]
Further, since $\frac{d(\lambda)}{d_p(\lambda)}$ is holomorphic in the strip, the last term is equal to $2\pi \log \left| \frac{d(iR)}{d_p(iR)} \right|$. It follows that
\[
\int_{\partial C'_R} \log^- \left| \frac{d(\lambda)}{d_p(\lambda)} \right| \ll \int_{\partial C'_R} \log^+ |d(\lambda)| + \int_{\partial C'_R} \log^- |d_p(\lambda)| - 2\pi \log \left| \frac{d(iR)}{d_p(iR)} \right|.
\]
Similarly, $\int_{\partial C'_R} \log^- |d_p(\lambda)| = \int_{\partial C'_R} \log^+ |d_p(\lambda)| - \int_{\partial C'_R} \log |d_p(\lambda)|$. Now, by Jensen’s formula, we have $2\pi \log |d_p(iR)| \leq \int_{\partial C'_R} \log |d_p(\lambda)|$. Plugging this into the above, we see
\[
\int_{\partial C'_R} \log^- \left| \frac{d(\lambda)}{d_p(\lambda)} \right| \ll \int_{\partial C'_R} \log^+ |d(\lambda)| + \int_{\partial C'_R} \log^+ |d_p(\lambda)| - 2\pi \log |d(iR)|.
\]
The above discussion gives the first two terms on the right are \( \ll R^{3Q} \). Tracing these inequalities backwards, we see

\[
\log^+ |m_J(\lambda, g)| \ll_{\Omega} R^{n+3} + R^{3Q} - \log |d(iR)|, \tag{5.8}
\]

for \( \lambda \in C_R \). Here, \( iR \) is the center of \( C_R \). By a linear fractional transformation which sends the disc \( C'_R \) to itself, but sends the center \( iR \) to any point in \( \tilde{C}_R = \{ \lambda : |\lambda - \lambda_0| \leq D_1/2 \} \) it is easy to see this same argument gives the same bound for \( \lambda \in C_R \), but the \( \log d \) term is evaluated at any point in \( \tilde{C}_R \). Thus, we need to show we can choose a point in \( \tilde{C}_R \) for which \( d \) evaluated at this point is not too small, in terms of \( R \). Once again, we will be able to do this, because we are plugging in \( \hat{\alpha}(\lambda)^{-1} \) into \( D_\alpha(s) \) which is of finite order globally.

Now, we have constructed a Selberg transform \( \hat{\alpha}(\lambda) \) so that \( \hat{\alpha}(\lambda)^{-1} \ll 1 + |\lambda|^Q \) for \( \lambda \) in the strip \( S \). Looking more closely at the specific function \( f_{q,c} \) which we used to find \( \alpha \), as in Proposition 5, we see that actually, asymptotically in \( S \) we have \( \hat{\alpha}(\lambda)^{-1} \gg |\lambda|^Q \). (These constants don’t have to be the same.) We will show the existence of a point \( iR_0 \in \tilde{C}_R \) so that \( |\hat{\alpha}(iR_0)^{-1} - \hat{\alpha}(\tau)^{-1}| \gg R^{Q-1} \) for all points \( \tau \) on the circle \( C_{R_0} = \{ \tau : |\tau - iR_0| = 1/R \} \). Recall that \( \hat{\alpha}(\lambda)^{-1} \) is a polynomial of degree \( Q \) over a product of sin terms: i.e., \( \hat{\alpha}(\lambda)^{-1} = \prod_{w} \sin_{w}(\lambda) \) (see the explicit form of \( f_{q,c} \) above). This was constructed so these sin terms do not vanish in the strip, and are easily seen to be uniformly bounded from above, and away from zero. For convenience, in this paragraph only, let us absorb the coefficient of \( \lambda^Q \) into the denominator, and call this resulting denominator \( g(\lambda) \). Let us call the resulting numerator (with leading coefficient one) \( f(\lambda) \). If \( \zeta \) is any complex number of length one, we have \( \tau \in C_{R_0} \) if \( \tau = iR_0 + \zeta/R \) for some such \( \zeta \). It follows that \( f(\tau) = f(iR_0) + O_{\zeta}(R^{Q-2}) \), where also \( |f(iR_0)| \) is asymptotically \( R^Q \). Suppose we have found a point \( iR_0 \) at which the derivative of \( g \) does not vanish and \( 1/R \) is sufficiently small so that we can write

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\[ 1/g(\tau) = 1/g(iR_0) + \zeta/R \cdot (1/g)'(iR_0) + O(R^{-2}). \]

Then,

\[ |\hat{\alpha}(iR_0)^{-1} - \hat{\alpha}(\tau)^{-1}| = \left| \frac{f(iR_0)}{g(iR_0)} - \frac{f(\tau)}{g(\tau)} \right| \]

\[ = \left| \frac{f(iR_0)}{g(iR_0)} - (f(iR_0) + O_\zeta(R^{Q-2})) \left( \frac{1}{g(iR_0)} + \frac{\zeta \cdot (1/g)'(iR_0)}{R} + O(R^{-2}) \right) \right| \]

\[ \gg |g'(iR_0)| R^{Q-1}. \]

All of this is easily attainable if \( g \) is periodic, and we get \( \gg R^{Q-1} \), since we only need one point, and then we just translate vertically. If however, this product of sines has different frequencies which are linearly independent (over \( \mathbb{Q} \)), by restricting to the imaginary axis, we have an \textit{almost periodic} function (on \( \mathbb{R} \)). Since each factor is periodic, we see we still have a Taylor estimate of \( 1/g(\tau) \) with the same \( O(R^{-2}) \) term. What we need is to find a point \( iR_0 \) so that \( g'(iR_0) \) is uniformly away from zero, essentially independent of \( R \). Since \( g \) is almost periodic, so is its derivative, and the result is clear.

Summing up the above results, we have a point \( iR_0 \in \tilde{C}_R \) with the property that \( \hat{\alpha}(\lambda)^{-1} \) sends the points \( \tau \) on the circle \( C_{R_0} \) to points that are a distance \( \gg R^{Q-1} \) from \( \hat{\alpha}(iR_0)^{-1} \). So, the image of the interior disc of \( C_{R_0} \) under \( \hat{\alpha}(\lambda)^{-1} \) contains a disc of radius \( \gg R^{Q-1} \). (This can be seen from the same property of the polynomial, and our explicit construction; asymptotically in the circle \( C_{R_0} \), there will be no winding from the \( \text{sin} \) terms.) Now, the image of \( S_{R+D_1} \) is clearly contained in a circle (centered at the origin) of radius \( \ll R^Q \). From before, the standard estimate gives there can be at most \( \ll R^{3Q} \) zeros of \( D_\alpha(s) \) in this larger circle. It follows if we put a ball of radius \( \frac{1}{R^{10}} \) around each zero of \( D_\alpha(s) \) in this larger circle, their total measure cannot exceed \( \ll R^Q \). Since the image of the interior disc of \( C_{R_0} \) contains a disc of measure
we see there are points \( \lambda \in \tilde{C}_R \) so that \( |\hat{\alpha}(\lambda)^{-1} - z_k| \gg \frac{1}{R^2Q}|z_1, z_2, \ldots \) is an enumeration of all the zeros of \( D_\alpha(s) \). Let us denote by \( \lambda_R \) any point in \( \tilde{C}_R \) with this property.

Now, by the finite order of \( D_\alpha(s) \), up to the exponential of a polynomial, by Hadamard’s Theorem we can write \( D_\alpha(s) = \prod_k E_3\left(\frac{s}{z_k}\right) \). Let us assume the zeros are listed so \( |z_k| \leq |z_{k+1}| \). Since \( D_\alpha(0) = 1 \), we have \( z_1 \neq 0 \). Here, \( E_3 \) is the usual Weierstrass factor: \( E_3(s) = (1-s) \cdot e^{(s^2 + s^3)} \). It is known that \( 1 - E_3(s) \) has a zero of order 4 at \( s = 0 \). More accurately, let us quote the estimate of Rudin [25] (page 301): for \( |s| \leq 1 \) we have \( |1 - E_3(s)| \leq |s|^4 \). For \( |s| \leq 1 \) it follows that \( |E_4(s)| \geq 1 - |s|^4 \).

We will use this estimate for \( |s| \leq \frac{1}{2} \). Let us assume \( \lambda_R \) is a point in \( \tilde{C}_R \) (with the above properties), and \( \lambda_R \) maps to \( s \). Then \( |s - z_k| \gg \frac{1}{R^2Q} \) for all \( k \). We will separate the above product into two products; over those \( k \) for which \( |s/z_k| \leq \frac{1}{2} \), and then its complement.

So, let us consider

\[
\prod_{|s/z_k| \leq \frac{1}{2}} |E_3(s/z_k)|.
\]

From the above estimate, this is

\[
\geq \prod_{|s/z_k| \leq \frac{1}{2}} (1 - |s/z_k|^4) = \exp \left( \sum_{|s/z_k| \leq \frac{1}{2}} \log(1 - |s/z_k|^4) \right)
\]

\[
\geq \exp \left( \sum_{|s/z_k| \leq \frac{1}{2}} -2|s/z_k|^4 \right),
\]

using \( \log(1 - r) \geq -2r \) if \( 0 \leq r \leq 1/2 \) (an obvious inequality from the series for log).
Moving the $|s|^4$ across the sum, it follows the expression above is

$$\geq \exp (-|s|^4 \sum_{|\frac{z_k}{s}| \leq \frac{1}{2}} |1/z_k|^4).$$

Since $D_\alpha$ is of finite order two, the above series converges, and we see there is a constant $c_4 > 0$ so that

$$\prod_{|\frac{z_k}{s}| \leq \frac{1}{2}} |E_3(s/z_k)| \geq \exp(-c_4 |s|^4). \quad (5.9)$$

Now we must estimate

$$\prod_{|\frac{z_k}{s}| > \frac{1}{2}} |E_3(s/z_k)|.$$ 

Recall that we still have $|s - z_k| \gg \frac{1}{R^2}$ for all $k$. If $|\frac{z_k}{s}| > \frac{1}{2}$, then each $|z_k| < 2|s|$. Since $|s| \ll R^Q$, the number of zeros $z_k$ for which this can be true is $\ll R^{3Q}$. Recall that $E_3(z) = (1 - s) \cdot e^{(z + \frac{z^2}{2} + \frac{z^3}{3})}$. With these two properties of $s$ and $z_k$ it follows that each term

$$|(1 - s/z_k)| \gg \frac{1}{R^{2Q}}.$$

Now, using the trivial estimate

$$\Re(s/z_k + \frac{(s/z_k)^2}{2} + \frac{(s/z_k)^3}{3}) \gg |s/z_k|^3$$

combined with the above, we have the existence of constants $c_5, c_6 > 0$ so that

$$\prod_{|\frac{z_k}{s}| > \frac{1}{2}} |E_3(s/z_k)| \gg \left(\frac{1}{R^{2Q}}\right)^{c_5 R^{3Q}} \exp(-c_6 |s|^3 \sum_{|\frac{z_k}{s}| > \frac{1}{2}} |1/z_k|^3).$$

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Once again, the above sum in the exponential is convergent. Let us just absorb this into $c_6$. Combining this with equation (5.9), we see

$$\prod_{\text{all } z_k} |E_3(s/z_k)| \gg \left( \frac{1}{R^{2Q}} \right)^{c_5 R^{3Q}} \cdot \exp(-c_4 |s|^4 - c_6 |s|^3).$$ (5.10)

Recall that $d(\lambda) = D_\alpha(\frac{1}{\alpha(\lambda)})$, and up to the exponential of a cubic polynomial, $D_\alpha(s) = \prod_k E_3(\frac{s}{z_k})$. It follows using the above inequality (5.10) that

$$\log(d(\lambda_R)) \gg -R^{3Q} - R^{3Q} \log(R^{2Q}) - R^{4Q}.$$ 

Returning to equation (5.8) and the immediate discussion, we have

$$\log^+ |m_J(\lambda, g)| \ll_{\Omega_G} R^{n_1+3} + R^{4Q}.$$ 

Let us record this result:

**Proposition 8** Let $\Omega_G$ be a compact set of $G$. With notations as above,

$$b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g) \ll_{\Omega_G} e^{\lambda |n_1 + |\lambda|^{4Q}}$$

for $\lambda$ in the strip $|\Re \lambda| \leq D_1$.

Recall $b(\lambda)$ was the Weierstrass product of all poles of coefficients of intertwining operators in the plane, and $d_p(\lambda)$ was the Weierstrass product built from $D_\alpha(\frac{1}{\alpha(\lambda)})$ in this strip.

Recall that $D_1$ was chosen so that $\{\Re \lambda \geq D_1/5\}$ was contained in the region of absolute convergence. Let $\Omega_G$ be a compact set of $G$. By Remark 2 of the Corollary to Lemma 24 of Harish-Chandra [10] we see $E(\lambda, \Phi, g)$ is of finite order one for $g \in \Omega_G$ and $\Re \lambda \geq D_1/2$. This is the trivial estimate from absolute convergence (but we...
have to stay uniformly away from the actual line of absolute convergence to use this estimate; that is why needed to be able to choose $D_1$. It follows that $b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g)$ will satisfy a finite order estimate with constant $n_1 + 3Q + 3$ for $\lambda$ in this region, for any $g \in \Omega_G$.

Now we use the functional equation, combined with absolute convergence. The functional equation can be written as:

$$E(\lambda, \Phi, g) = E(s\lambda, C(s|\lambda)\Phi, g).$$

(See Wong [35] page 163.) Let us recall some specific information. We suppose $P_1$ is associate to $P_j$. Here $s$ is the nontrivial element of the relative Weyl group, and $C(s|\lambda)$ is the intertwining operator from $^0L^2(M \cap \Gamma\backslash M, \chi)$ to $^0L^2(M_j \cap \Gamma\backslash M_j, ^s\chi)$. If $\{\Phi_p\}_p$ and $\{\Psi_{j,s,n}\}_n$ are their respective bases, and if $C(s|\lambda)$ has the matrix form $[\vartheta(j, s, n; p|\lambda)]$ in these bases, then if we write our cusp form as $\Phi = \sum c_p \Phi_p$, we have

$$C(s|\lambda)(\Phi) = \sum_n \theta(j, s, n|\lambda)\Psi_{j,s,n},$$

where $\theta(j, s, n|\lambda) = \sum_p c_p \vartheta(j, s, n; p|\lambda)$. We desire a finite order bound for $b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g)$ for $\Re \lambda \leq -D_1/2$, and $g \in \Omega_g$.

It follows that

$$b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g) = d_p(\lambda) \cdot E(s\lambda, b(\lambda) \cdot \sum_n \theta(j, s, n|\lambda)\Psi_{j,s,n}, g).$$

Now, $b(\lambda)$ was constructed to be the Weierstrass product consisting of all the poles of the $\theta(j, s, n|\lambda)$. It follows that each $b(\lambda)\theta(j, s, n|\lambda)$ is entire of finite order $n_1 + 2$ in the plane. If $\Re \lambda \leq -D_1/2$, then $s\lambda$ is in the region of absolute convergence (and uniformly away from the line of convergence). So using the same estimate from the Remark of Harish-Chandra [10] (as above) pointwise in $\lambda$ for $\lambda$ in this half plane, we
see \( b(\lambda) \cdot d_p(\lambda) \cdot E(\lambda, \Phi, g) \) satisfies a finite order bound with constant \( n_1 + 3Q + 3 \) in this left half plane.

Combining these last three paragraphs with Proposition 8 we have the main result of this thesis.

**Theorem 2** Let \( \Omega_G \) be a compact set of \( G \). With notation as above,

\[
E(\lambda, \Phi, g) = \frac{F(\lambda, g)}{b(\lambda)d_p(\lambda)},
\]

where \( b(\lambda)d_p(\lambda) \) is an entire function of order \( n_1 + 3Q + 2 \). Further \( F(\lambda, g) \) is entire for each \( g \in \Omega_G \) and satisfies the estimate

\[
F(\lambda, g) \ll \Omega_G e^{\mid\lambda\mid^{Q}}
\]

for all \( \lambda \in \mathbb{C} \), if \( Q > n_1 + 2 \).

Finally, let us give a few remarks on the computability of this bound; namely \( Q \). Recall, we are inverting the explicit transform \( f_{q,c} \) from above. If \( q \) is sufficiently large, let \( \alpha_{q,c} \) denote the compactly supported point-pair invariant, the inverse spherical transform of \( f_{q,c} \). We saw \( Q \leq q \cdot |W| \). Further, \( f_{q,c} \) was constructed so the support of \( \alpha_{q,c} \) did not depend on \( q \) or \( c \). So, we just need to increase \( q \) to increase the number of derivatives \( \alpha_{q,c} \) will have, with uniform support. In our previous Poisson-sum type estimates, we needed to integrate by parts functions of the form \( \alpha(hng) \) in the \( n \) variable, for \( h \) and \( g \) in a compact set, for \( n \) in different unipotent subgroups. From this uniformity, we see we get an estimate of the form \( k_\alpha(h, g) \ll \eta(s_g)^p-N \), as before.

We can take larger \( N \) by increasing \( q \). (The specifics of this relationship between \( N \) and \( q \) of course also depends on the group. Unfortunately, computations become unsightly very quickly.) In the above integrals, we need \( K_\alpha H_J \) to be bounded for \( \lambda \) in a strip, and uniformly for \( g \in \Gamma\backslash G \). However, the growth of \( H_J \) depends on which
strip we want. So, we need to first choose our strip, with much overlap (as above),
and then choose \( f_{q,c} \) with \( q \) sufficiently large so all integrals are absolutely convergent.
We see everything is computable.

### 5.6 The case of several cusps

In this section we will extend the main result to the case that \( \Gamma \backslash G \) has several
cusps. Actually, this is a simple exercise in reduction theory. Wong [35] (chapter
6) has worked out the relevant combinatorics, and we just need to verify that our
estimates still hold. Recall, our group \( G \) is connected, real, split over \( \mathbb{Q} \), and semi-
simple; it is either a classical group or of type \( G_2 \).

Let \( \Xi \subset \mathbb{Q} \) consist of one representative from each double coset \( \Gamma \backslash \mathbb{Q} / \mathbb{Z} \). From
reduction theory (see [2]) we know that \( \Xi \) is actually finite. Further, there is a Siegel
domain \( \mathcal{S}_0 \) so that \( G = \Gamma \Xi \mathcal{S}_0 \). In particular, \( \Xi \mathcal{S}_0 \supset \mathcal{F} \), a fundamental domain for
\( \Gamma \backslash G \). Let us write \( \Xi = \{ \zeta_\nu \} \) where \( \nu \) is a finite index. If \( \mathcal{F} \) is a function on \( \Gamma \backslash G \), we will
say it is slowly increasing if there are constants \( C_1, C_2 \) so that \( |f(\zeta_\nu g)| \leq C_1 \eta(s_g)^{C_2} \)
for \( g \in \mathcal{S} \) and all \( \nu \). If \( \mathcal{F} \) is any function on \( G \), let us denote by \( \zeta_\nu \mathcal{F} \) left translation;
\( \zeta_\nu \mathcal{F}(g) = \mathcal{F}(\zeta_\nu g) \). Further, if \( H \) is any subgroup of \( G \), let us denote \( \zeta_\nu H = \zeta_\nu H \zeta_\nu^{-1} \).

Then, \( \mathcal{F} \in L_p(\Gamma \backslash G) \) implies \( \zeta_\nu \mathcal{F} \in L_p(\zeta_\nu^{-1} \Gamma \backslash G) \) with equal \( p \) norms. This follows
from the fact that if \( \mathcal{F} \) is a fundamental domain for \( \Gamma \backslash G \), so is \( \mathcal{F} \zeta_\nu \) since the action
of \( \Gamma \) is on the left.

Similar to before, let us consider for \( f \in L^1(\Gamma \backslash G) \)

\[
(\alpha * \zeta_\nu f - \alpha * (\zeta_\nu f')(g)
\]

for \( g \in \mathcal{S}_0 \). However, if we apply the same analysis as above (basically the com-
putations from equations (4.1) through (4.4)) to \( \zeta_\nu f \) as a function on \( (\zeta_\nu^{-1} \Gamma \backslash G \) we
see

\[ (\alpha * \zeta f - \alpha * (\zeta f)^j)(g) \ll \frac{\beta_j(s_g) \beta(s_g)}{\alpha_j(s_g)^N} \| \zeta_f \|_{L^1(\Gamma^{-1}\backslash G)} \]

where there might be an additional constant coming from \( \nu \). Since there are only finitely many \( \nu \) this won’t concern us. Now, from the above paragraph we can use the same bound with the \( L^1 \) norm of \( \zeta_f \) replaced with \( \| f \|_{L^1(\Gamma \backslash G)} \).

In the case of several cusps, we will have to modify our compact operator \( K_\alpha \) to include more parabolic constant terms. So, our cut-off functions will be more complicated. To start, recall the partition of unity \( \{ \delta_r \} \) from before. Given \( \nu \) from the above index set, let us define \( e_{\nu} = (\zeta^{-1}\delta_r) \) as a function on \( G \). Let \( \mathcal{S} \) be a Siegel domain large enough so that its interior \( \mathcal{S}^0 \) contains \( \mathcal{S}_0 \). Wong [35] (pages 175, 176) then constructs a partition of unity \( \omega_\nu \) on \( \Xi \mathcal{S}^0 \) so that \( \omega_\nu(g) = 0 \) if \( g \notin \zeta_\nu \mathcal{S}^0 \). Now, let us define \( \delta_{r\nu} = \omega_\nu e_{r\nu} \) as a function on \( G \). Since \( \omega_\nu \) is supported on \( \Xi \mathcal{S}^0 \), off of this set \( \delta_{r\nu} \) will actually vanish. For notation, let \( P_{r\nu} = \zeta P_r \), and \( f^{r\nu} \) denote the constant term of \( f \) along \( P_{r\nu} \). For \( f \in L^1(\Gamma \backslash G) \) let us consider

\[ \alpha * f - \sum_{j\nu} \delta_{j\nu} \alpha * f^{j\nu}(g) \]

for \( g \in \Xi \mathcal{S}_0 \).

Wong [35] considers this sum (page 177), but is more interested in an \( L^2 \) estimate. If we look at this argument, if we fix \( j \), since \( \omega_\nu(g) = 0 \) if \( g \notin \zeta_\nu \mathcal{S}^0 \), for an individual \( \nu \), we only need to consider \( g \) in this set. Then, we are in almost the same situation as above. Basically, we have a sum of terms of the form

\[ \omega_\nu(\zeta_\nu g') \delta_j(g')(\alpha * \zeta_f - \alpha * (\zeta f)^j)(g'), \]

where \( p \) and \( N \) are as above. Here, \( g = \zeta_\nu g' \) so \( g' \in \mathcal{S}^0 \). Now, if \( \delta_j(g') = 1 \), then
\[ \alpha_j(s_{y'}) = \eta(s_{y'}) \] and it follows the above is

\[ \ll \eta(s_{\tilde{y}})^{p-N} \| f \|_{L^1(\Gamma \backslash G)}. \]

Otherwise, if \( 0 < \delta_j(g') < 1 \) we are in the small set, which is taken care of (with maybe a larger constant) by the construction of the \( \{ \delta_r \} \), exactly as above. It follows that

\[ \alpha * f - \sum_{j\nu} \delta_{j\nu} \alpha * f^{j\nu} (g) \ll \eta(s_{\tilde{y}})^{p-N} \| f \|_{L^1(\Gamma \backslash G)}. \]

Here, \( \eta(s_{\tilde{y}}) \) will denote the maximum of the numbers \( \eta(s_{y'}) \). We are only taking the maximum over the \( \nu \) for which \( g = \zeta_{\nu} g' \) and \( g' \in S^0 \).

For \( f \in L^2(\Gamma \backslash G) \), put

\[ K_\alpha f = \left[ \alpha * f - \sum_{j\nu} \delta_{j\nu} \alpha * f^{j\nu} \right]. \]

Then, as before, \( K_\alpha \) can be expressed as an integral operator where the kernel \( k_\alpha \) not only is bounded, but satisfies

\[ |k_\alpha(h,g)| \ll \eta(s_{\tilde{y}})^{p-N}. \]

From the same argument as before, \( K_\alpha \) is a compact operator on \( L^2(\Gamma \backslash G) \).

Now, our Eisenstien series, \( E \), is built from a cusp form on the Levi of \( P_1 \). If \( P_{j\nu} \) is not associate to \( P_1 \), then the constant term of \( E \) along \( P_{j\nu} \) is zero. If \( P_{j\nu} \) is associate to \( P_1 \), then we have Langlands’ formula:

\[ E^{j\nu}(\lambda, g) = \sum_{s,n} \theta(j\nu, s, n|\lambda)e^{(s\lambda + \rho_{j\nu})\log a_{j\nu}}\Psi_{j\nu,s,n}(m_{j\nu}), \]
where the notation parallels the previous work (see Wong [35] page 180). Let

\[ I_{j\nu,s,n}(\lambda, g) = \delta_{j\nu}(g)e^{(s\lambda+\rho_{j\nu})\log a_{j\nu}}\psi_{j\nu,s,n}(m_{j\nu}) \]

for \( g \in \Xi S^0 \). Finally, we define

\[ H_{j\nu,s,n}(\lambda, g) = [I_{j\nu,s,n}(\lambda, \cdot)]_F. \]

We need to show that \( H_{j\nu,s,n} \) is slowly increasing for \( \lambda \) in a strip.

Similar to before, Wong [35] (pages 181,182), if \( \lambda \) is in a strip, its real part is compact. Wong’s Proposition 6.4.1 gives the growth of \( I_{j\nu,s,n}(\lambda, g) \) is then bounded in terms of the \( \alpha_r(A(\zeta'_{\nu} g)) \). Here, \( \zeta'_{\nu} \) is the compact part of \( \zeta_{\nu} \). We need the automorphic extension (with the above \( F \)) of this, \( H_{j\nu,s,n}(\lambda, g) \). For this, we need to relate \( \alpha_r(A(\zeta'_{\nu_1} \gamma^{-1}\zeta_{\nu_2} g)) \) for each \( r \) to \( \eta(A(g)) \) for \( g \in S_0 \) so there exists \( \gamma \) and \( g' \in S_0 \) so that \( \gamma\zeta'_{\nu_1} g' = \zeta_{\nu_2} g \). What saves us here is the Siegel property: \( \{ \gamma | \gamma \Xi S_0 \cap \Xi S_0 \neq \emptyset \} \) is finite. So, \( \zeta'_{\nu_1} \gamma^{-1}\zeta_{\nu_2} \) is contained in a finite set. We now cite Harish-Chandra’s estimate ([10] Lemma 21a). We take for the compact set all the relevant \( \zeta'_{\nu_1} \gamma^{-1}\zeta_{\nu_2} \). Since \( K \) is on the right, not the left, the inequality is reversed with a \( +\infty \). The gives a sufficient relation, and proves \( H_{j\nu,s,n} \) is slowly increasing for \( \lambda \) in a strip.

If we use a multi-index \( \bar{J} \) for \( \{ j\nu, s, n \} \), then our Fredholm equation becomes

\[ (K_\alpha - \hat{\alpha}(\lambda))F_{\bar{J}} = -K_\alpha H_{\bar{J}} \]

for \( \lambda \) off the spectrum of \( K_\alpha \). Now, we know the kernel of \( K_\alpha \) is slowly decreasing; \( |k_\alpha(h, g)| \ll \eta(s_{\bar{g}})^{p-N} \), accounting for multiple cusps. Now \( H_{\bar{J}} \) is slowly increasing for \( \lambda \) in a strip, the estimate accounting for multiple cusps, as well. It follow that if \( N \) is sufficiently large (if \( \alpha \) has enough derivatives) that \( K_\alpha H_{\bar{J}} \) is uniformly bounded for \( g \in \Gamma \setminus G \) and \( \lambda \) in a strip. Using (5.2) and the immediate discussion, it follows
that the numerator of $F_{J^*}^{**}$ satisfies a finite order bound for $\lambda$ in a strip. If we put $F_{J^*} = F_{J^*}^{**} + H_{J^*}$, then it follows that

$$E(\lambda, g) = \sum J \theta(J|\lambda) F_{J^*}(\lambda, g),$$

for $\lambda$ in a strip. Now, the denominator of $F_{J^*}$ is under control in the strip (as before), so citing Müller [23] to handle the $\theta(J|\lambda)$ in all of $\mathbb{C}$, the rest of the argument is the same.

### 5.7 The Adelic picture

We need to transfer the main result to Eisenstein series on adelic groups. These will still be constructed from a cusp form on a Levi factor of a maximal parabolic. Since one can only differentiate an adelic function from an archimedean component, it is not difficult to believe that most of the general analytic properties carry over easily; so this section will be short, and we will refer the reader to several places. The actual computation of the intertwining operator (at least the unramified part) into an Euler product involving the eigenvalues of the Hecke algebra the cusp form produces (for each prime) is the subject of Langlands [19]. Indeed, this work was the birth of the $L$-group which is crucial for functoriality.

We assume $G$ is a split, connected, semi-simple linear algebraic group over $\mathbb{Q}$, which is either a classical group, or of type $G_2$. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$. Let $G = G(\mathbb{A})$. Similar to before, $\mathcal{B}$ will be a Borel subgroup of $G$ with split component $\mathbb{A}$ and unipotent radical $U$, all over $\mathbb{Q}$. $P$ will be a maximal $\mathbb{Q}$-parabolic containing $B$. We thus can write $P = N\mathbb{A}_1M$ with Levi component $M$. From this, we can write $G = PK$ for an obvious maximal compact $K$. Let $a_1 = Hom(X(M)_{\mathbb{Q}}, \mathbb{R})$ be the real Lie algebra of $\mathbb{A}_1$. Here $X(M)_{\mathbb{Q}}$ is the group of $\mathbb{Q}$-rational characters of $M$. 

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Then, there exists a homomorphism $H_P : M \to a_1$ incorporating all of the $p$-adic and archimedean imbeddings $a_{1_p} \hookrightarrow a_1$. (See §1 of Shahidi [32].) We will use $H_P$ for the adelic analogue of $e^{(\lambda+\rho)\log(a_1)}$. Let $\rho_P$ denote half the sum of positive $\mathbb{Q}$-roots generating $N$. We assume $\alpha$ is the positive simple root deleted to construct $M$. We can then identify $\lambda \in \mathbb{C}$, the complex dual of $a_1$ with $\lambda \cdot \frac{\rho_P}{<\rho_P, \alpha>}$. (See Shahidi [32]. This turns out to be the correct normalization.)

Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $M$. (See the Corvallis paper of Borel and Jacquet [3].) Since we are using spherical inversion above, we are forced to assume if $\phi$ is a function in the space of $\pi$, that it is invariant on the right under $\xi = \xi_\infty \otimes \xi_L$ in the global Hecke algebra. More specifically, $\xi_\infty$ must be the characteristic function of $K_\infty$, and $L$ is some open compact subgroup of $K^f$, the finite part of $K$. Further, we need to assume $\pi$ is generic; i.e., each $\pi_v$ can be realized in a space of functions $W_v$ satisfying $W_v(um) = \chi_v(u)W_v(m)$ where $\chi = \otimes \chi_v$ is a generic character of $U(\mathbb{Q}) \backslash U$, if $m \in M_v$ and $u \in (U \cap M)_v$. (See §3 of Shahidi [32], or the papers [28] [30] [31] [33] of Shahidi.)

We extend $\phi$ to a function $\tilde{\phi}$ on $G$ as in Shahidi [28] (which is similar to page 26 of Harish-Chandra [10]; recall, our $K$-type is trivial). Now, put $\Phi_\lambda(g) = \tilde{\phi}(g)\exp(<\lambda + \rho_P, H_P(g)>)$. Then the adelic Eisenstein series is defined as (for $g \in G$ and $P$ denoting $P$)

$$E(\lambda, \tilde{\phi}, g, P) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi_\lambda(\gamma g)$$

(cf. Langlands [19] and also [28]). For each $g$, this series converges for $\Re \lambda$ sufficiently large (cf. [19]), is meromorphic in all of $\mathbb{C}$ and has a functional equation (cf. [20], in particular the appendix on adeles).

In particular, the functional equation comes from integrating $E(\lambda, \tilde{\phi}, ng, P)$ over $n \in N'(\mathbb{Q}) \backslash N'(\mathbb{A})$ where $N'(\mathbb{A})$ is the unipotent radical of a maximal parabolic $P'$
associate to \( P \). Now, \( \phi \) is an automorphic cusp form on \( M \). More specifically, \( \phi \) is invariant on the left by \( M(Q) \), is invariant on the right by \( \xi = \xi_\infty \otimes \xi_L \) (and for us, \( \xi_\infty \) as above), is annihilated by an ideal \( J \) of \( Z(M_\infty) \), and is still a cusp form. We further assume \( J \) is maximal, so \( \phi \) is an eigenfunction for \( Z(M_\infty) \). In the notation of Borel and Jacquet \([3]\), \( \phi \in {}^0A(\xi_\infty \otimes \xi_L, J, (K \cap M)_\infty) \). It follows from general principles that taking the constant term of this Eisenstein series along \( \mathcal{P}' \) gives rise to a global intertwining operator

\[
M(\lambda): {}^0A(\xi_\infty \otimes \xi_L, J, (K \cap M)_\infty) \to {}^0A(\xi_\infty \otimes \xi'_L, J', (K \cap M')_\infty).
\]

(Obviously, some data has been conjugated.)

Now, we can write \( M(\mathbb{A}) = M(Q) \cdot C \cdot M_\infty \cdot L \), where \( L \) is some compact open subgroup of \( M' \), and \( C \) is a finite set of points in \( M(\mathbb{A}) \) (cf. \([3]\)). For each \( c \in C \), put

\[
\Gamma_c = M(Q) \cap (M_\infty \times c \cdot L \cdot c^{-1}).
\]

Each \( \Gamma_c \) is an arithmetic discrete subgroup of \( M_\infty \). Further, we have an isomorphism

\[
M(\mathbb{Q}) \backslash M(\mathbb{A})/L \simeq \prod_c (\Gamma_c \backslash M(\mathbb{R})).
\]

which easily gives

\[
{}^0A(\xi_\infty \otimes \xi_L, J, (K \cap M)_\infty) \simeq \bigoplus_c {}^0A(\Gamma_c, \xi_\infty, J, (K \cap M)_\infty).
\]

Here, the terms of the right hand side are automorphic (by \( \Gamma_c \)) cusp forms on \( M_\infty = M(\mathbb{R}) \). It follows that the global intertwining operator, \( M(\lambda) \) above retains all the analytic properties from before. In particular, from Müller \([23]\) we have any matrix coefficient of \( M(\lambda) \) is a meromorphic function of finite order.
In the Fredholm theory above, we must now replace $\Gamma \backslash G$ by $G(\mathbb{Q}) \backslash G(\mathbb{A})$. Further, our non-smooth compactly supported, $K_\infty$ bi-invariant function $\alpha_\infty$ is replaced with $\alpha = \prod_v \alpha_v$ where $\alpha_v$ equals the characteristic function of $G(Z_v)$ for almost all finite $v$. Here $Z_v$ is the ring of $p$-adic integers in $\mathbb{Q}_p$ (we are still over $\mathbb{Q}$), and $\alpha_\infty$ is exactly the type of function as before. We need to show the adelic integral operator $K_\alpha$ is compact on $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$; actually, we need the kernel of this integral operator $k_\alpha$ to be bounded on the product. For this, we see Godement’s estimates [9] are actually adelic. We are still over $\mathbb{Q}$, and in this paper, one still uses Siegel sets. It follows that everything goes through, as before. More precisely, in the Selberg transform of $\alpha = \prod_v \alpha_v$, the convolution of $\Phi_\lambda(g)$ with $\alpha$ can be reduced to a convolution on the Levi $M$ of $\phi$ with a function $\alpha_\lambda$ (see page 30 of Harish-Chandra [10]). From the isomorphism above for $\phi$ and $M$, we have a finite number of $\Gamma_c \backslash M(\mathbb{R})$. However, the cusp form we can take on each $\Gamma_c \backslash M(\mathbb{R})$ must have trivial $K$-type, and the same eigenvalues under $J$. Since the Selberg transform only uses these properties, and doesn’t see the discrete subgroup, we see (under this isomorphism) we can take the Selberg transform of the adelic $\alpha$ to be any of these $\hat{\alpha}_\infty(\lambda)$. Thus, we can apply the theory of Fredholm determinants to invert $K_\alpha - \hat{\alpha}_\infty(\lambda)$. Using spherical inversion, and Müller [23], Theorem 2 follows in the adelic case, as before.
Chapter 6

Langlands-Shahidi L-functions

In this final chapter, we will show how Theorem 2 can be used (along with Theorem 1) to prove the result of Gelbart and Shahidi [7] in a handful of examples; boundedness in vertical strips of certain $L$-functions. The point is, we have removed the dependence of a functional equation in any delicate way.

Let $G$ be a split, connected, semi-simple, linear algebraic group over $\mathbb{Q}$ (either a classical group, or of type $G_2$), and $G = G(\mathbb{A})$ where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Suppose $P$ is a maximal parabolic subgroup of $G$. Further, suppose $M = M(\mathbb{A})$ and $N$ are the Levi factor and unipotent radical (respectively) of $P$ in the usual manner. Let $\pi$ be a globally generic cuspidal automorphic representation of $M$. We construct an adelic Eisenstein series $E(s, \tilde{\phi}, g, P)$, as above. ($s \in \mathbb{C}$ will replace $\lambda$. Further, there is a natural normalization of $s$ which we shall use, due to Shahidi; for instance see [32].) Now, $M$ acts on $N$ by adjoint action. This gives an action on the dual side. Decompose the adjoint action $r$, of $L^M$ on the complex Lie algebra $L^n$ of $L^N$ as $r = \oplus_{i=1}^{m} r_i$. Let $L(s, \pi, r_i)$ be the completed automorphic $L$-function attached to $\pi$ and $r_i$, where the local factors at ramified places are defined in Shahidi [34].

Now, $\pi = \otimes \pi_v$. If $v = \infty$, using Casselman’s subrepresentation theorem, $\pi_v \hookrightarrow I_M(\nu_0, \eta_0)$, a principal series representation of $M$. Now, if $I(s, \pi)$ is the unitarily nor-
malized parabolic induction from \( P \) to \( G \), then \( I(s, \pi) \leftrightarrow I(\nu(s), \eta_0) \), a principal series of \( G \). Here \( \nu(s) \) is a line in the \( \rho_P \) direction that intersects \( \nu_0 \). (See Shahidi [7] or Gelbart and Shahidi [29].) Essentially, the function \( \Phi_s(g) = \tilde{\phi}(g) \exp(<s + \rho_P, H_P(g)>) \) from the last section is in the space \( I(s, \pi) \). We have enough information to build a Whittaker function. Recall from before the function \( E_V(g, D, \chi, \lambda; \zeta) \) from the section on Whittaker functions. This was a local integral of a function in the principal series twisted by an additive character. Here \( D \) was a representation of the maximal compact subgroup, \( \chi \) was a unitary character of the split component, \( \lambda \) was in the complex dual of the split component, and \( \zeta \) was an additive character on the full unipotent subgroup \( V \) (opposite \( U \)). Let \( \sigma_\infty \) denote the \( K \)-type of \( \pi_\infty \). Easily, \( \chi_\infty \) can be found which matches \( \sigma_\infty \) on the intersection of the split component with the maximal compact. (From the injections above, \( \chi_\infty \) must be related to \( \eta_0 \).) Now, we are assuming \( \pi \) is globally generic. So there is a generic additive character \( \psi \) on \( U(Q) \backslash U(A) \) whose restriction to \( U(A) \cap M \) is the generic character of \( \pi \). For \( v = \infty \) let us define \( W_{v,s}(e) \) to be the Whittaker function \( E_V(e, \sigma_\infty, \chi_\infty, \nu(s); \psi_\infty) \). (Technically, this is not what we will want, only since the action of the maximal compact subgroup is on the left, not the right. This gives no essential difference. We will take \( W_{v,s}(e) \) to be the corresponding analytically continued Jacquet integral at \( \nu(s) \), where the actions are of that of today (\( K \) is on the right). The result (and proof) of Theorem 1 is clear.)

Let \( E_\psi(s, \tilde{\phi}, g, P) \) be the \( \psi \)-Fourier coefficient of \( E(s, \tilde{\phi}, g, P) \) along \( U(Q) \backslash U(A) \). Let \( S \) be the finite set of primes consisting of \( \infty \), plus any primes \( v \) that ramify in \( \pi_v, G, \) or \( \psi_v \). Then, by Shahidi’s computations [30] [32], we have

\[
E_\psi(s, \tilde{\phi}, e, P) = \prod_{v \in S} W_v(s, e_v) \cdot \prod_{i=1}^m L_S(1 + is, \pi, \tilde{r}_i)^{-1},
\]

where \( L_S \) means the partial \( L \)-function, and \( \tilde{r}_i \) is the contragredient of \( r_i \).
Recall, we are restricting $G$ to be a classical group, or of type $G_2$. Further, we assume our automorphic cuspidal representation $\pi$ has trivial $K$-type. Then, since $U(\mathbb{Q}) \backslash U(\mathbb{A})$ is a compact set, the adelic version of Theorem 2 gives that $E_\psi(s, \tilde{\phi}, e, P)$ is a ratio of entire functions of computable finite order. For $v = \infty$, Theorem 1 clearly gives $W_v(s, e_v)$ an entire function of finite order. For the finite ramified $v$, quoting Casselman and Shalika [4], for finite $v \in S$, $W_v(s, e_v)$ is a rational function in $p^{-s}$ ($p$ the prime $v$), so clearly meromorphic of finite order. Combining all of these results, we have $\prod_{i=1}^m L_S(1 + is, \pi, \tilde{r}_i)^{-1}$ is a ratio of entire finite order functions; i.e., meromorphic with computable finite order. Now, Shahidi [34] has defined the local $L$-functions for ramified places. Since one can get only $\Gamma$ factors, or rational functions in $p^{-s}$, it follows that $\prod_{i=1}^m L(1 + is, \pi, \tilde{r}_i)$ is meromorphic with computable finite order. Let us record this result as a corollary to Theorems 1 and 2.

**Corollary 1** Suppose $M(\mathbb{A}_\mathbb{Q})$ embeds as the Levi component of a maximal parabolic subgroup of a group $G$, which is a classical group, or of type $G_2$. If $\pi$ is an automorphic cuspidal representation of $M$ with trivial $K$-type, then

$$\prod_{i=1}^m L(1 + is, \pi, \tilde{r}_i)$$

is a ratio of entire functions of computable finite order.

With a local assumption on $\pi$ (Assumption 2.1 of [7]), Gelbart and Shahidi proved by induction each $L(s, \pi, \tilde{r}_i)$ (known to be meromorphic) has only a finite number of poles (and all real). We will use this assumption as well. If we knew an individual $L(s, \pi, \tilde{r}_i)$ were a ratio of finite order functions, this would give boundedness in vertical strips, away from the poles. As described in [7], this is an easy consequence of the absolute convergence of $L(s, \pi, \tilde{r}_i)$ combined with the functional equation (cf. Shahidi [34]), and an application of Phragmen-Lindelöf’s theorem. We would thus have the result for $L(s, \pi, \tilde{r}_m)$ if we knew each $L(s, \pi, \tilde{r}_i)$ were meromorphic of finite order for
each $i < m$.

**Example 1** This is example (ii) of Langlands [19].

Let $G = SL_{n+1}(\mathbb{R})$, defined and split over $\mathbb{Q}$. Deleting the root at the end of the Dynkin diagram gives the Levi $M \simeq GL_n$. The decomposition of the action is just $m = 1$, with $r$ being the standard representation of $GL_n$. Let $\pi$ be a generic cuspidal automorphic representation of $GL_n(\mathbb{A})$ with trivial $K$-type. Then its contragredient $\tilde{\pi}$ will also be a generic cuspidal automorphic representation of $GL_n(\mathbb{A})$. We take $\tilde{\pi}$ as our cusp form on $M$. Since $m = 1$, the above Corollary with the immediate argument gives $L(s, \pi)$ is bounded in vertical strips.

**Example 2** This is example (iii) of Langlands [19].

Let $G = SL_{n_1+n_2}(\mathbb{R})$, defined and split over $\mathbb{Q}$. Deleting a root near the middle of the Dynkin diagram gives a Levi $M \simeq \{ (g_1, g_2) \in GL_{n_1} \times GL_{n_2} | \det(g_1g_2) = 1 \}$. Let $\pi_1$ and $\pi_2$ be generic cuspidal automorphic representations of $GL_{n_1}(\mathbb{A})$ and $GL_{n_2}(\mathbb{A})$ respectively, both with trivial $K$-types, and trivial central characters. We take $\tilde{\pi}_1 \times \pi_2$ as our cusp form on $M$. The decomposition of the action is $m = 1$ and $r$ agrees with the tensor product of the standard representation of $GL_{n_1}$ with the contragredient of the standard representation of $GL_{n_2}$ on this cusp form. (Note the extra condition of trivial central character.) Since $m = 1$, by the Corollary, we have $L(s, \pi_1 \times \pi_2)$ is bounded in vertical strips, away from its poles.

**Example 3** This is example (iv) of Langlands [19].

Let $G = SO_{2n+1}(\mathbb{R})$, defined and split over $\mathbb{Q}$. Deleting the short root at the end of the Dynkin diagram gives a Levi $M \simeq GL_n$. The decomposition of the action is $m = 1$ with $r$ being the symmetric square representation of the standard representation of $GL_n$. Let $\pi$ be a generic cuspidal automorphic representation of $GL_n(\mathbb{A})$, with trivial $K$-type. Put $\tilde{\pi}$ as our cusp form on $M$. Since $m = 1$, by the Corollary, we have
Example 4 This is example (vi) of Langlands [19].

Let $G = Sp_{2n}({\mathbb{R}})$, defined and split over $\mathbb{Q}$. Deleting the long root at the end of the Dynkin diagram gives a Levi $M \simeq GL_n$. The decomposition of the action is $m = 2$ with $r_1$ being the standard representation of $GL_n$, and $r_2$ being the exterior square representation of the standard representation. Let $\pi$ be a generic cuspidal automorphic representation of $GL_n(\mathbb{A})$, with trivial $K$-type. Put $\tilde{\pi}$ as our cusp form on $M$. Consequently, the product in the above Corollary is $L(1+s, \pi) \cdot L(1+2s, \wedge^2(\pi))$.

From Example 1, we have the first term is a ratio of finite order functions, and actually bounded in vertical strips away from it poles. From the discussion immediately following the Corollary, this gives $L(s, \wedge^2(\pi))$ is a ratio of entire finite order functions, and consequently bounded in vertical strips, away from its poles.

Example 5 This is example (xv) of Langlands [19].

Let $G$ be a real group of type $G_2$ defined and split over $\mathbb{Q}$. Delete the short root in the Dynkin diagram. Then, the corresponding Levi $M \simeq GL_2$. Further, the decomposition of the action gives $m = 2$ with $r_1$ the adjoint cube representation of $GL_2$, and $r_2$ is the one dimensional representation of $GL_2$ (i.e., the exterior square of the standard representation; central character). Here, the adjoint cube is just the symmetric cube twisted by the inverse of the central character (cf. Shahidi [33]). Langlands [19] calculated the weight of $r_1$, but the computation of Shahidi [33] is necessary to pin down $r_1$ explicitly. Let $\pi$ be a generic cuspidal automorphic representation of $GL_2(\mathbb{A})$, with trivial $K$-type. If $\omega_\pi$ denotes the central character of $\pi$, let us define $\pi_1 = \omega_\pi \otimes \pi$. Let $\pi_2$ denote the contragredient of $\pi_1$. Let us take $\pi_2$ as the cusp form on the Levi. It follows the product in the above Corollary is $L(1+s, Sym^3(\pi)) \cdot L(1+2s, \omega_\pi^3)$. Now the second term here is just a Dirichlet $L$-function, which is clearly a ratio of entire order one functions. As before, from the
discussion immediately after the above Corollary, it follows $L(s, Sym^3(\pi))$ is a ratio of entire computable finite order functions, and consequently is bounded in vertical strips away from its poles.
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